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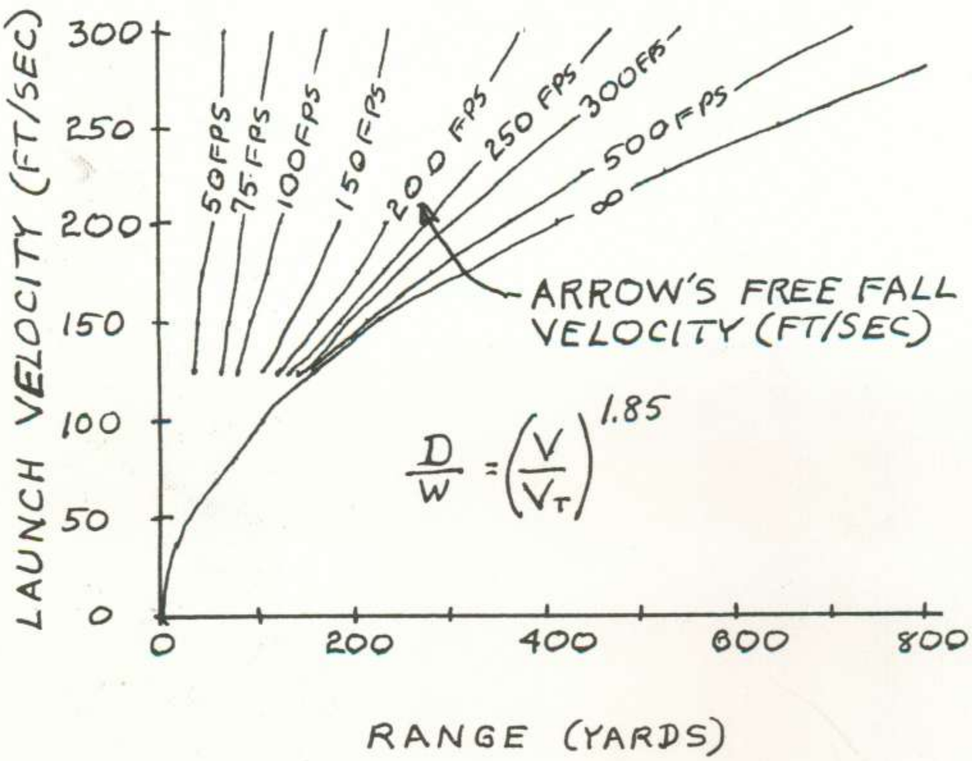
PHYSICAL LAWS OF ARCHERY

by Thomas L. Leistor, P.E.



THIRD EDITION

PHYSICAL LAWS OF ARCHERY

by Thomas L. Liston, P.E.

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INTRODUCTION

Thomas L. Liston, Mechanical Engineer, is one of those engineers who wants to know what the laws of nature say about whatever it is in which he happens to be interested. When he took up archery, he was surprised to find very little published on archery engineering. The last authoritative book on the subject came out in the 1940s. So, being curious, Mr. Liston set out to compute virtually everything that can be computed about archery. Although he is a competent target archer and bowhunter, he makes a living as a consulting mechanical engineer specializing in heating, ventilation and air conditioning. His engineering education includes a Bachelor of Science in Mechanical Engineering from the University of California, Berkeley.

Liston simply set out to put numbers on every aspect of archery which was readily computed. He drew upon classic statics, dynamics, fluid flow, aerodynamics, strength of materials and statistics. In doing so, he basically wrote a primer. The simple things, such as draw work, hysteresis, virtual mass, and arrow energy are only simple when someone explains them in a simple way. The tough things, such as bow shape versus energy storage and recovery, and such as the dynamics of arrow vibration during launch, were taken only as far as simple analysis would permit. The intermediate things, such as the calculation of arrow friction and its effect on trajectories and energy loss were taken as far as the author's mastery of applicable math would permit.

The engineer-archer will doubtlessly find it interesting to see which laws of nature explain which feature of archery, and to see which things are too complicated to readily explain. The author left specific challenges to interested engineers to solve particular problems which he, the author, could not solve. The inquisitive non-engineer archer will learn a lot about engineering in reading this book. Or, if not interested in engineering, an archer may simply read the narrative for conclusions.

For the first time in history, graphs showing how far an arrow will go when fired at any particular velocity are published. The corresponding launch velocities required to achieve particular maximum ranges are also published for the first time. These charts are the compilation of months of computer runs.

Thomas L. Liston, P.E.

Thomas L. Liston, P.E.
Mechanical Engineer
January 1988

<u>About the Second Edition</u>:

Chapter 13, "Penetration", has been greatly expanded, mainly due to the urging of Dan Quillian, President of Archery Traditions. November 1989.

<u>About the Third Edition</u>

Chapter 3 on Virtual Mass has been updated to incorporate the latest concepts developed by Norb Mullaney. June 1990.

PHYSICAL LAWS OF ARCHERY

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The archer does work when he draws his bow. The bow does work on the arrow when the shot is fired. Understanding the relationships between the work done by the archer when drawing and the work done by the bow when shooting is important. If the bow and arrow combination were 100% efficient, all of the archer's effort would translate into arrow energy. As it turns out, somewhere in the neighborhood of 75% of the archer's work does propel the arrow. The lost energy goes partly into internal friction within the bow, which is called "hysteresis". This often amounts to around 8%. The remainder of the lost energy is found in those parts of the bow which are still in motion at the moment of separation of arrow from string.

To measure the amount of work done by the archer, a spring scale and an arrow marked in inches of draw is needed. See photo. The pull in pounds at each inch of draw can then be measured. When the pull forces at each inch of draw are added up, they tell the amount of work done. The dimensions are "inch-pounds". Engineers prefer to talk of "foot-pounds", so the inch-pounds are divided by 12 to get foot-pounds. When a person lifts a pound of butter up one foot high, he imparts one foot-pound of work to the butter.

RECURVE &/or LONGBOWS

Figure 1-1 shows the data collected for a recurve bow. A blank form for the reader's use is provided for use with his own longbow or recurve. Figure 1-2 shows a plot of the same data. A blank form upon which to plot is also provided.

A recurve bow (and/or a longbow) requires ever-increasing force as the draw length is increased. The let-down forces are a little less than the draw forces, but it requires fancy instrumentation to measure the difference. The forces measured when using a spring scale and slowly changing lengths is somewhere between draw force and let-down force. The form provided for recurve &/or longbows ignores the differences.

The area under the curve represents the total energy the archer has expended while drawing the bow. Instructions with the form for data gathering tell how to compute this area. The reader who is comfortable with computing areas under curves will devise his own procedures. Were the force-draw curve of a long bow or recurve exactly a straight line, a single computation would suffice. An excellent approximation can be made by drawing a straight line through the data, aligning the straight line to make the area under it equal to the area under the actual line. See the straight line in Figure 1-2.

COMPOUND BOWS

Compound bows are better at accepting energy than are recurve bows. The force required to draw a compound builds up more rapidly. The required force reaches a peak during mid-draw and then diminishes. The "let-off" is typically 50% or so, which diminishes the energy being stored somewhat. The net result, though, is that --- for the same peak draw force --- a compound accepts more energy than a recurve. See Figures 1-3 & 1-4 for data on the author's compound hunting bow.

When measuring peak draw force on a spring scale, the difference during prompt draw and during prompt let-down can be noticed and measured. The difference represents the energy lost to internal friction, called "hysteresis". When slowly drawing a compound on a spring scale, the maximum draw force falls between the prompt draw and prompt let-down forces. For instance, the author's hunting bow registers 73½# during prompt draw, 69# during prompt let-down, and 72# during gradual measurement. Because these differences can be seen on an ordinary spring scale, it is possible to compute the bow's hysteresis. For a recurve bow, this separate identification of the bow's hysteresis cannot be done without resorting to sophisticated instrumentation.

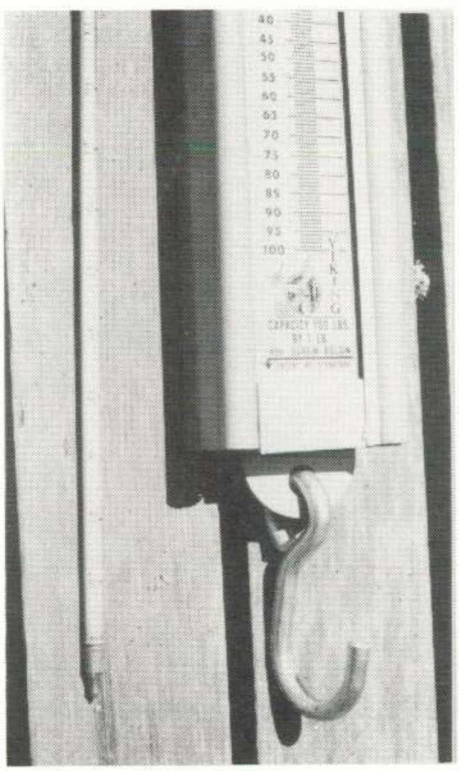
The form for compound bows makes use of the measurability of hysteresis. The bow's "virtual mass" (which is defined in later chapters) is computed differently for the compound bow. Input energy (which is slightly higher than measured statically) and recoverable energy (which is slightly lower than measured statically).

PEAK DRAW:

The general shape of a bow's draw-force curve stays about the same regardless of peak draw force. The curve for a 60# long bow and for a 30# long bow look exactly alike except for the numbers. Similarly, adjustable-force compound bows have similar looking curves regardless of power setting. The result is that the amount of energy stored is a constant multiplier of the peak draw force. The fraction of energy lost to hysteresis is almost constant, regardless of force setting. Similarly, as will be explained in the chapter on "virtual mass", the fraction of energy left behind when the arrow is launched changes hardly at all.

Consequently, for a given adjustable bow, the amount of energy stored is a direct function of the peak draw force. Arrow velocity (assuming the same arrow is shot) is a square root function of the energy. Thus, adjusting a bow upwards from 40# to 44# would increase the stored energy 10%. The square root of 1.10 is 1.049 and thus a 10% increase of stored energy would yield a 4.9% increase in arrow velocity.

A word of caution, however. That rule applies to small increases. For large increases, such as from 40# to 60#, different arrows would be recommended. For instance, for a 40 pound compound bow shooting a 30" long hunting arrow, Easton recommends a size 2016 weighing 507 grains. At 60 pounds the recommendation is a size 2117 weighing 551 grains. Thus some of the speed increase would be sacrificed for a heavier arrow. For how to calculate exactly what the speed change would be, see the chapter entitled "virtual mass".



Above left: Son Alex Liston drawing 55# Martin Cougar target compound.

Above right: Author Tom Liston drawing 70# Hoyt Rambo hunting compound.

Left: Spring scale and arrow marked at 1" intervals.

RECURVE OR LONGBOW ENERGY CALCULATION FORM

Bow Identification & Set-up Data:

Make: Ben Pearson . Model: #304 Recurve take-down .
Serial nr. or other I.D.: Fiberglas, Mfg. about 1965. .
Brace height: 9" . Weight of bow's mass: 2# .
Upper limb, turns out (adjustable bows only): — .
Lower limb, turns out (adjustable bows only): — .
Date data taken: 3-23-86 . Bow owned by: Alex Liston .

Spring Scale Data:
Make: Hansen . Model: 8910 . Range: 0 - 100 # .
Date calibrated: New '86 . Corrections None known .

Bow Test Data:

1. Draw force = $48\frac{1}{2}$ # at draw length of $29\frac{1}{4}$" inches.
2. Draw lengths & corresponding draw weights:

0"		10"	4	20"	28	30"	50(Ignore)
1"		11"	7	21"	30	31"	
2"		12"	10	22"	32	32"	
3"		13"	13	23"	34	33"	
4"		14"	15	24"	36	34"	
5"		15"	17	25"	38	35"	
6"		16"	19	26"	40	36"	
7"		17"	21	27"	43	37"	
8"		18"	24	28"	45	38"	
9"	ϕ	19"	25	29"	48 (Use $35\frac{1}{4}$)	39"	

3. Col. sums: ϕ + 155 + $361\frac{1}{4}$ + 0 * =
   * Add per rules; do not add area beyond draw length.
4. Sum of 4 columns in line 7: $516\frac{1}{4}$ inch-lbs.
5. Stored energy: (Divide line 8 by 12: 43.0 foot-pounds.
6. Stored energy per peak draw force (Line 5 divided by Line 1): 0.89 foot-pounds per pound.

Virtual Mass Calculation (If arrow's weight & speed known.)

7. Arrow's weight 470 grains. For 1916 x 29"
8. Arrow's velocity 148 ft/sec
9. Arrow's energy $= \frac{1}{2}mV^2 =$
   line 7 x line 8 x line 8 divided by 450,800 = 22.8 ft-lbs.
10. Ratio, arrow energy to input energy (line 9/line 5): 0.53 .
11. Ratio, bow's virtual mass to arrow's weight = (1 - line 10)/line 10 = 0.883 .
12. Bow's virtual mass = line 11 x line 7 = 415 grains.

Figure 1-1 ... Recurve Force-Draw Data

FORCE - DRAW PLOT

BOW: MAKE: BEN PEARSON MODEL: # 304 TAKE-DOWN RECURVE

SERIAL OR I.D.: FIBERGLASS. MFG ABOUT 1965 OWNER: ALEX LISTON

SETUP: UPPER LIMB: — LOWER LIMB: —

SPRING SCALE I.D.: HANSEN # 8910, 0 → 100# New '86. No calib. data.

ARROW DATA: Enter identification, weight, velocity.

1ST ARROW: 1916 × 29", field point, 470 grains, 148 fps, 29¼" draw

2ND ARROW: 2117 × 30", " ", 543 grains, 151 fps, 30¼" draw.

COMPUTED DATA: Virtual mass = 394 grains

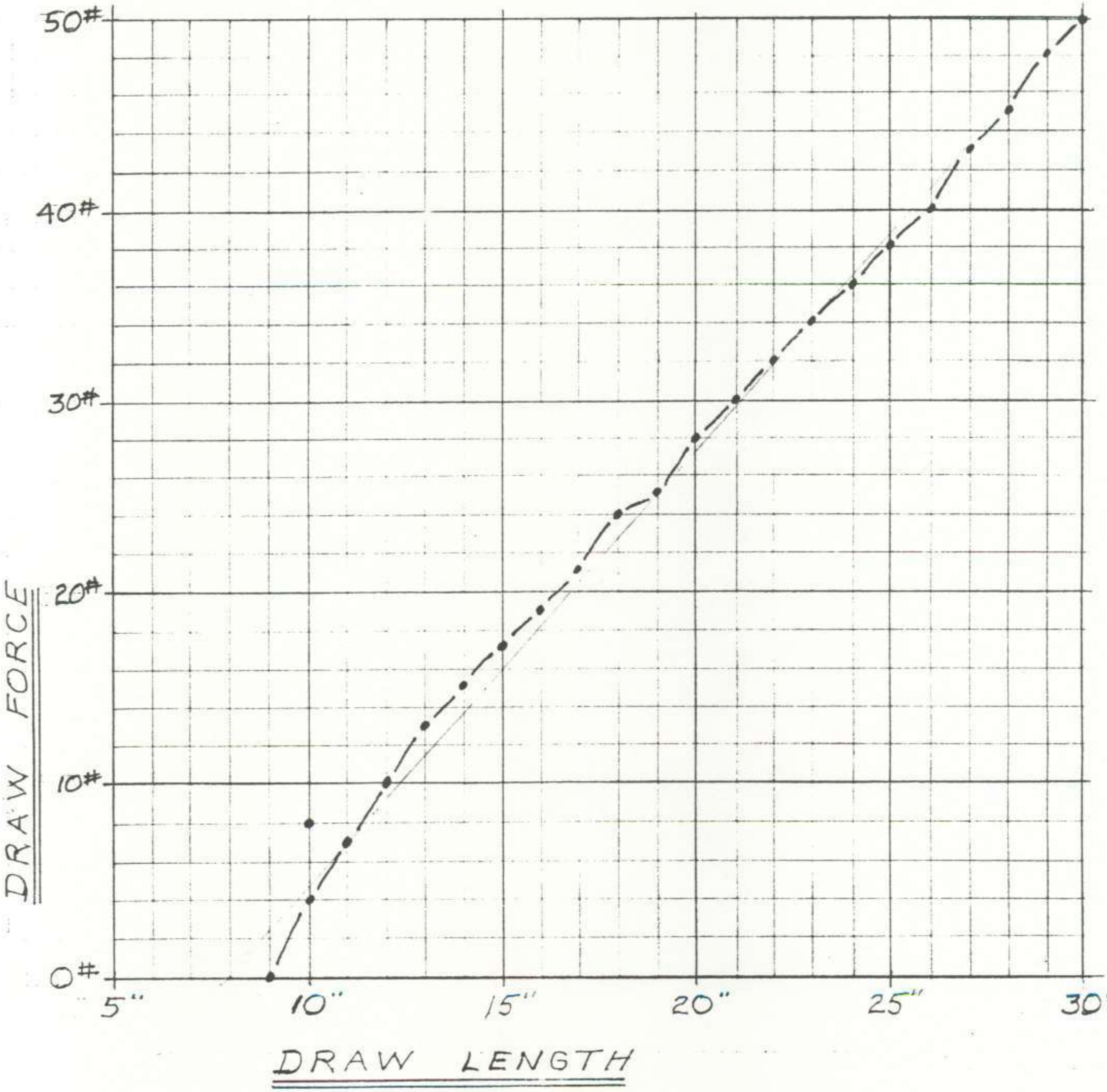


Figure 1-2 ... Recurve Force-Draw Plot

COMPOUND BOW ENERGY CALCULATION FORM

Bow Identification & Set-up Data:
Make: Hoyt . Model: RAMBO . Wheel: ROUND .
Serial nr. or other I.D.: 55/70 NOMINAL RATING .
Brace height: ______ . Weight of bow's mass: 7# .
Draw length (Use length at which force is minimum.): 29½ .
Upper limb, turns out (adjustable bows only): Ø .
Lower limb, turns out (adjustable bows only): Ø .
Date data taken: 2/16/87 . Bow owned by: Tom Liston .

Spring Scale Data:
Make: HANSEN . Model: 8910 . Range: 0 → 100# .
Date of last calibration: New '86 . Corrections needed: None known .

Bow Test Data:

1. Peak draw force during prompt draw: 73½# .
2. Peak draw force during prompt let-down: 69 # .
3. Hysteresis loss (Subtract line 2 from line 1.): 4½# .
4. Fraction loss to hysteresis (Divide line 3 by line 1.): 0.061 .
5. Percentage loss to hysteresis (Multiply line 4 by 100.): 6.1%.
6. Draw lengths & corresponding draw weights:

0"		10"		20"	67½	30"	35½
1"		11"	Ø	21"	70	31"	
2"		12"	7½	22"	72	32"	
3"		13"	14½	23"	71	33"	
4"		14"	21½	24"	67½	34"	
5"		15"	28½	25"	61½	35"	
6"		16"	38	26"	57	36"	
7"		17"	47½	27"	50½	37"	
8"		18"	55	28"	44	38"	
9"		19"	62	29"	35	39"	

7. Col. sums: Ø 274½ 594 Ø *
   * Add per rules; do not add area beyond draw length.
8. Sum of 4 columns in line 7: 868½ inch-lbs.
9. Stored energy: (Divide line 8 by 12: 72.4 foot-pounds.
10. Recoverable energy (Line 9 x line 2 divide by line 6's peak draw:
    72.4 × (69/72) = 69.4 foot-pounds.
11. Stored energy per peak draw force (Line 10 divided by Line 1):
    69.4/73.5 = 0.944 foot-pounds per pound.

Virtual Mass Calculation (Calculate if arrow's weight & velocity are known.)

12. Arrow's weight 543 grains, as measured by scale .
13. Arrow's velocity 214 ft/sec, as determined by meter .
14. Arrow's energy = $\frac{1}{2}mV^2$ = line 12 x line 13 x line 13 divided by 450,800 =
    543 × 214 × 214 ÷ 450,800 = 55.2 foot-pounds.
15. Ratio, arrow energy to recoverable energy (line 14 / line 10): 0.795 .
16. Ratio, bow's virtual mass to arrow's weight (1 - line 15)/line 15 0.26 .
17. Bow's virtual mass = line 16 x line 12 = 140 grains.

Figure 1-3 ... Compound Force-Draw Data

FORCE - DRAW PLOT

BOW: MAKE: HOYT MODEL: RAMBOW

SERIAL OR I.D.: 55/70 NOMINAL OWNER: TOM LISTON

SETUP: UPPER LIMB: Φ LOWER LIMB: Φ

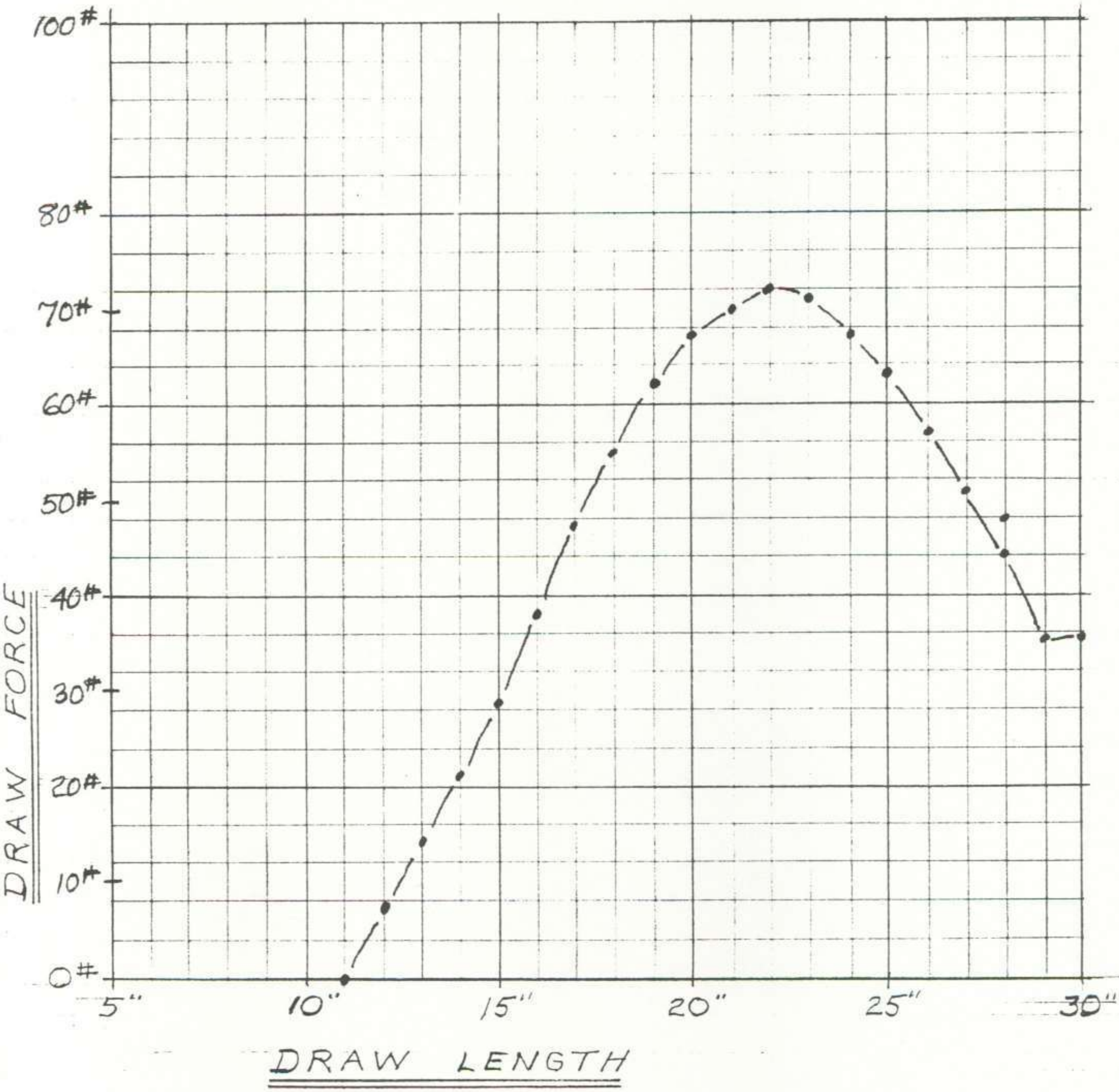
SPRING SCALE I.D.: HANSENO ⇁100# MODEL 8910

ARROW DATA: Enter identification, weight, velocity.

1ST ARROW: 2117 x 30" FIELD TIP, 543 GRAINS, 214 FT/SEC.

2ND ARROW:

COMPUTED DATA: VIRTUAL MASS = 140 GRAINS. 72.4 ft-# in; ft-# OUT



FORM 1-1 ... FORCE - DRAW PLOT JAN. 1988

Figure 1-4 ... Compound Force-Draw Plot

RECURVE OR LONGBOW ENERGY CALCULATION FORM

Bow Identification & Set-up Data:

Make: ________________. Model: ________________________________.
Serial nr. or other I.D.: ____.
Brace height: ____________. Weight of bow's mass: ____________.
Upper limb, turns out (adjustable bows only): _______.
Lower limb, turns out (adjustable bows only): _______.
Date data taken: ____________. Bow owned by: ________________.

Spring Scale Data:

Make: _____________. Model: _____________. Range: ____________.
Date calibrated: ____________. Corrections ______________________.

Bow Test Data:

1. Draw force = _________# at draw length of ________ inches.
2. Draw lengths & corresponding draw weights:

0"	________	10"	________	20"	________	30"	________
1"	________	11"	________	21"	________	31"	________
2"	________	12"	________	22"	________	32"	________
3"	________	13"	________	23"	________	33"	________
4"	________	14"	________	24"	________	34"	________
5"	________	15"	________	25"	________	35"	________
6"	________	16"	________	26"	________	36"	________
7"	________	17"	________	27"	________	37"	________
8"	________	18"	________	28"	________	38"	________
9"	________	19"	________	29"	________	39"	________

3. Col. sums: ________ ________ ________ ________*
   * Add per rules; do not add area beyond draw length.
4. Sum of 4 columns in line 7: ______________________ inch-lbs.
5. Stored energy: (Divide line 8 by 12: __________ foot-pounds.
6. Stored energy per peak draw force (Line 5 divided by Line 1): ________________________________ foot-pounds per pound.

Virtual Mass Calculation (If arrow's weight & speed known.)

7. Arrow's weight ___________ grains.
8. Arrow's velocity ______ ft/sec.
9. Arrow's energy = $\frac{1}{2}mV^2$ =
   line 7 x line 8 x line 8 divided by 450,800 = _____ ft-lbs.
10. Ratio, arrow energy to input energy (line 9/line 5): ______.
11. Ratio, bow's virtual mass to arrow's weight = (1 - line 10)/line 10 = ____________.
12. Bow's virtual mass = line 11 x line 7 = ____________ grains.

COMPOUND BOW ENERGY CALCULATION FORM

Bow Identification & Set-up Data:

Make: ______________. Model: ______________. Wheel: ________________.

Serial nr. or other I.D.: ____.

Brace height: ___________. Weight of bow's mass: __________________.

Draw length (Use length at which force is minimum.): ______________.

Upper limb, turns out (adjustable bows only): ______.

Lower limb, turns out (adjustable bows only): ______.

Date data taken: __________. Bow owned by: ________________________.

Spring Scale Data:

Make: ______________. Model: ______________. Range: ________________.

Date of last calibration: __________. Corrections needed: __________.

Bow Test Data:

1. Peak draw force during prompt draw: ____________________________.
2. Peak draw force during prompt let-down: ________________________.
3. Hysteresis loss (Subtract line 2 from line 1.): ________________.
4. Fraction loss to hysteresis (Divide line 3 by line 1.): ________.
5. Percentage loss to hysteresis (Multiply line 4 by 100.): _______.
6. Draw lengths & corresponding draw weights:

0" ________	10" ________	20" ________	30" ________
1" ________	11" ________	21" ________	31" ________
2" ________	12" ________	22" ________	32" ________
3" ________	13" ________	23" ________	33" ________
4" ________	14" ________	24" ________	34" ________
5" ________	15" ________	25" ________	35" ________
6" ________	16" ________	26" ________	36" ________
7" ________	17" ________	27" ________	37" ________
8" ________	18" ________	28" ________	38" ________
9" ________	19" ________	29" ________	39" ________

7. Col. sums: ________ ________ ________ ________*
   * Add per rules; do not add area beyond draw length.
8. Sum of 4 columns in line 7: ______________________ inch-lbs.
9. Stored energy: (Divide line 8 by 12: __________________ foot-pounds.
10. Recoverable energy (Line 9 x line 2 divide by line 6's peak draw:
    ____ foot-pounds.
11. Stored energy per peak draw force (Line 10 divided by Line 1):
    __________________________________ foot-pounds per pound.

Virtual Mass Calculation (Calculate if arrow's weight & velocity are known.)

12. Arrow's weight __________ grains, as measured by ______________.
13. Arrow's velocity ______ ft/sec, as determined by ______________.
14. Arrow's energy = $\frac{1}{2}mV^2$ = line 12 x line 13 x line 13 divided by 450,800 =
    ____ foot-pounds.
15. Ratio, arrow energy to recoverable energy (line 14 / line 10): ______.
16. Ratio, bow's virtual mass to arrow's weight (1 - line 15)/line 15 ____.
17. Bow's virtual mass = line 16 x line 12 = ____________________ grains.

Rules for addition of first & last increments of draw:

A. First non-zero measurement is used without correction.
B. Note shooting draw length. All measurements which are 1/2" or more shorter than draw length are added without correction.
C. Compute work done drawing last fraction of an inch by computing what fraction of an inch is involved and what average force is involved.

Example:

Draw length = 29¼"

Force at 28" = 45# (= average from 27½" to 28½").
Force at 29" = 48# (= average from 28½" to 29½").
Force at 30" = 50#

Final draw length = 29¼" - 28½" = 3/4"
Average force over final 3/4" = 46.5# @ 27½" + ¼ x (50-48)# = 47.0#.
Energy = force x length = 47.0# x 3/4" = 35¼ inch-pounds.

Note that computing the correct final force is complicated but important with long bows. With compound bows, the final force stays constant and thus need not be computed, as the final force is in the "bottom of the valley".

With either type bow, the final length of draw must be computed as illustrated in the example. A half inch difference in draw will impart a noticeably different velocity to an arrow. The difference is much greater for a long bow than for a compound bow.

To be technically correct, the first fraction of an inch of draw should be corrected in the same way as the final. The forces are so small during the first inch of draw that the failure to be exact causes no significant error.

FORCE - DRAW PLOT

BOW: MAKE: ________________ MODEL: ________________

SERIAL OR I.D.: ________________________ OWNER: __________

SETUP: UPPER LIMB: __________ LOWER LIMB: __________

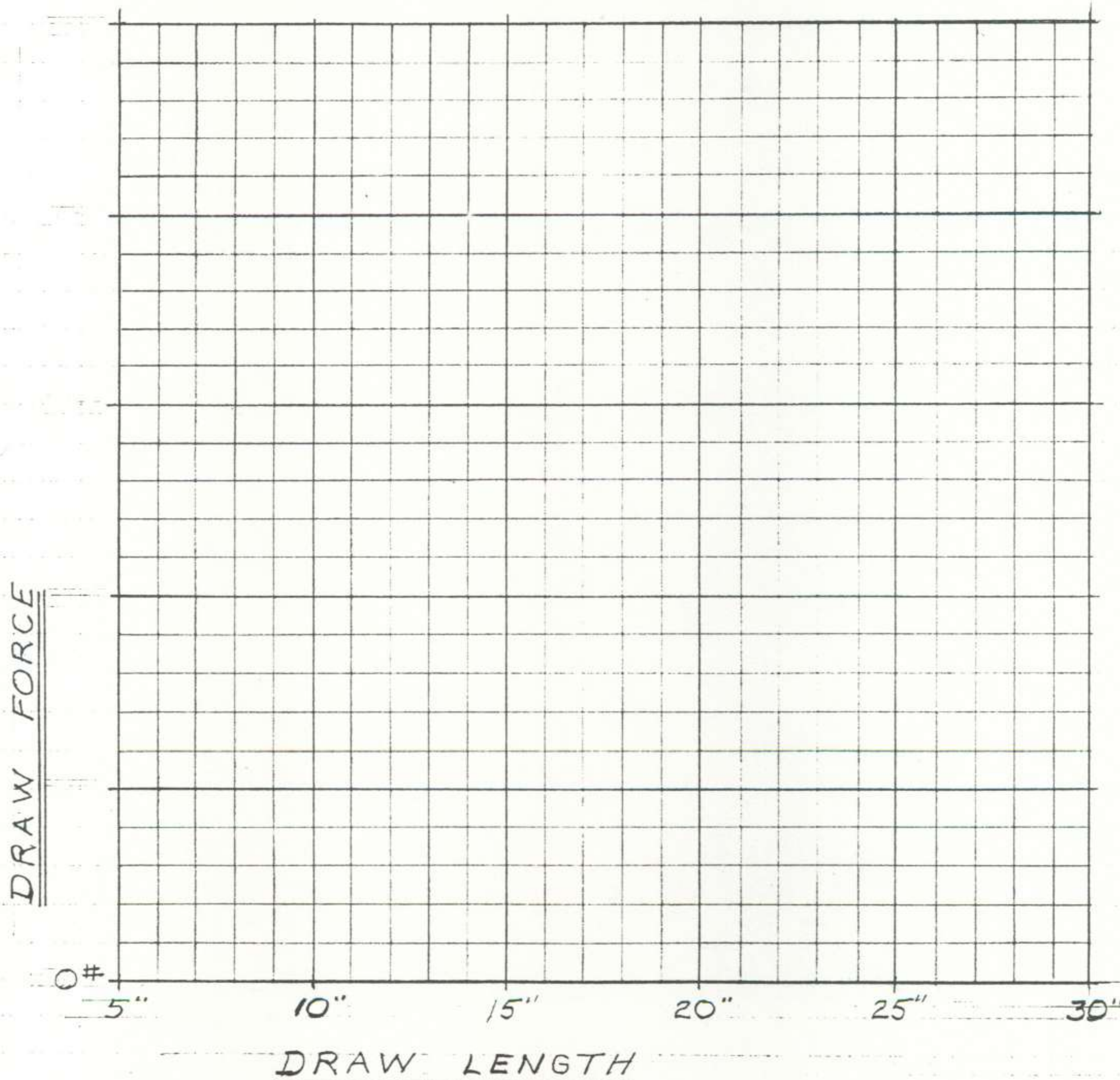
SPRING SCALE I.D.: ______________________________

ARROW DATA: Enter identification, weight, velocity.

1ST ARROW: ______________________________

2ND ARROW: ______________________________

COMPUTED DATA: ______________________________



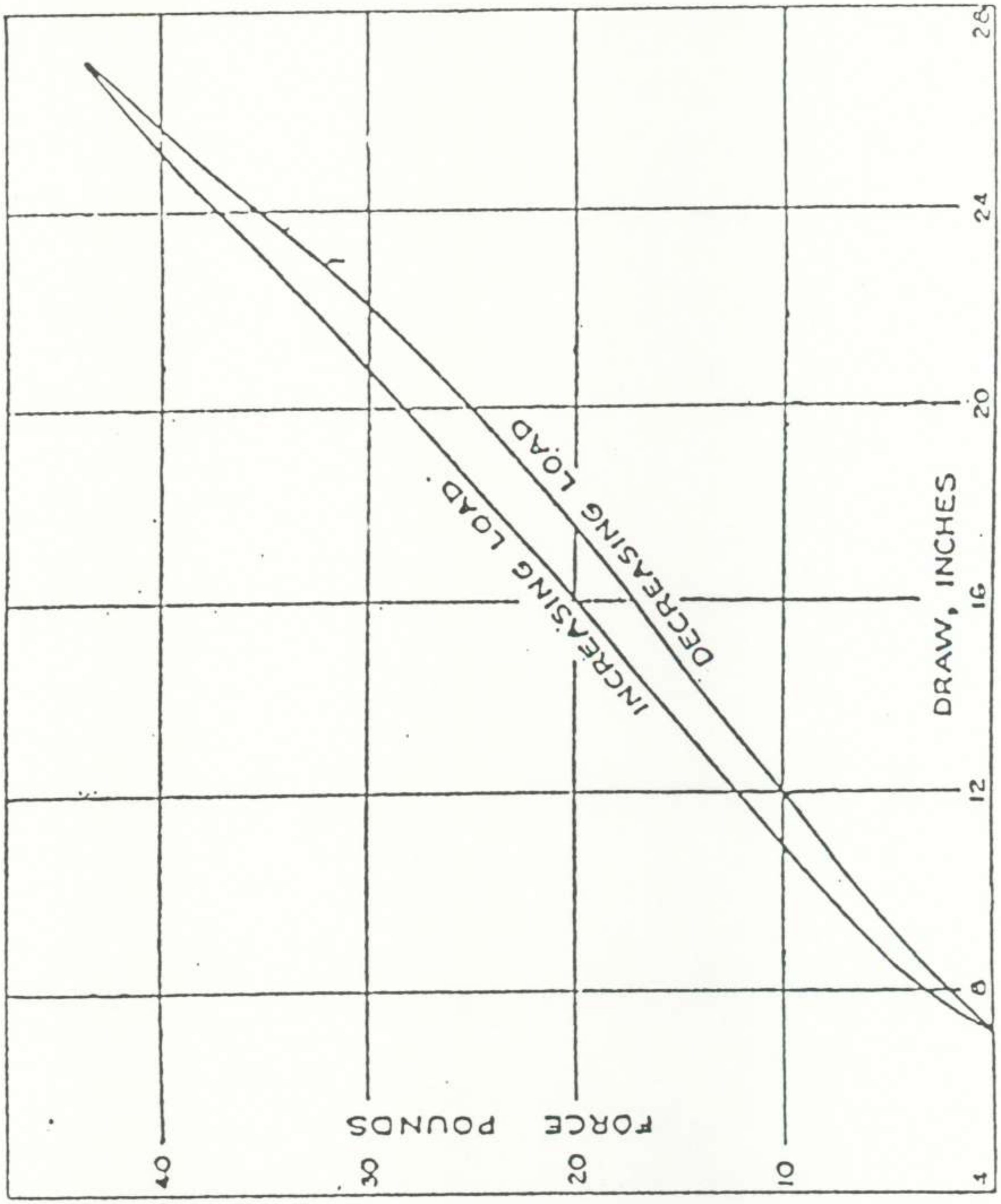
Chapter 2 ... BOW ENERGY LOSSES: HYSTERESIS

The author notes that when promptly drawing his compound on a spring scale, the bow developes a peak of 73½# when drawing, but developes only about 69# when letting it back down promptly. From this I conclude that this particular bow can deliver no more than 69/73½ths of the energy put into it, even when given all the time in the world to do it. This hints at an energy loss of 4½/73½ or 6.1%, but I place no confidence in the accuracy of such a crude measurement except to note that the concept of an energy loss is indeed noticeable on a spring scale. Note in Figure 1-3 that peak pull is 72# when gradually increased, which is between 69# and 73½#.

The loss of energy between draw and let-down is due to internal friction within the bow, which is called "hysteresis". Oneida Labs, Incorporated's published data on their Oneida Eagle Bow dated 11-11-82 shows a loss of 7.6% to 8.2%. Loss of 5% to 20% were reported by Paul E. Klopsteg in "Physics of Bow and Arrows published in the American Journal of Physics in August 1943. He was talking about bamboo bows. He measured the draw and let-down pulls using a movie camera taking 16 frames per second. Figure 2-1 is a reproduction of Mr. Klopsteg's results.

The data shown for the author's bow does not illustrate hysteresis, because it was taken so slowly that the difference between draw and let-down was totally muddied. Were the same bow tested as did Mr. Klopsteg, using a camera at 16 frames per second, there would be two, not one, curves. The draw curve would have slightly higher valves than shown, and the let-down curve would have slightly lower values.

Efficiency is what engineers tend to adulate. A bow's peak efficiency is equal to 100% less the hysteresis. Thus, peak efficiencies for bows vary between about 80% to 95%. The joker, however, is that the only way for a bow to deliver peak efficiency is to let down slowly. If bows were used to shoot crow bars, their peak efficiencies would tell us what we want to know. When firing arrows, however, bows' efficiencies are much less than their peak efficiencies. The reason for the drop of efficiency from peak when shooting arrows is that the bow imparts velocity not only to the arrow but also to various parts of the bow. The chapter on "virtual mass" goes into this aspect in depth. The bottom line, however, is that the virtual mass of the bow when compared to the arrow it is shooting, is a much more meaningful number than is efficiency.



Hysteresis loop for a bow of laminated bamboo, showing an energy loss of about 12 percent.

Figure 2-1 ... Klopsteg's Hysteresis Illustration

Virtual Mass

The efficiency of a bow is the ratio of energy imparted to the arrow compared to the amount of work done drawing the bow. As already mentioned, hysteresis accounts for a loss of from 5% to 20%. Actual efficiencies vary a great deal, and the same bow will have efficiencies which vary with the arrow shot. A "dry-fired" bow, for instance, has an efficiency of zero because there is no arrow to accept any energy. The same bow when shooting an extremely heavy arrow, such as a crowbar, will have an efficiency almost equal to 100% less the hysteresis. Thus the inherent potential efficiency of a bow is (100% - 5%) to (100% -20%) = 95% to 80%.

When shooting an ordinary arrow, parts of the bow are still in motion at the moment the arrow leaves the string. For instance, that part of the string which was in contact with the arrow is traveling at the same speed as the arrow at the moment of separation. The limbs of the bow are still in motion at the moment of separation. Accounting for all the energy remaining within the bow and it's components at the moment of separation would be quite complicated if each component were examined individually. Fortunately, an easier method was developed by Paul E. Klopsteg in about 1940.

The concept of "virtual mass" was developed as an easy way to account for all the energy remaining with the bow at the moment of releasing an arrow. It makes no attempt to explain exactly where and how that energy is; rather, it simply computes a number to describe the energy. "Virtual mass" is defined as being that mass which, were it traveling at the same speed as the arrow, would account for the lost energy.

An example: A bow has a 10% hysteresis loss. It propels an arrow at such speed as to account for 60% of the bow's input work. The remaining 30% is presumed to be still in the bow at the moment of release. Since the arrow has 60% and the bow has 30%, the virtual mass of the bow is said to be half of the mass of the arrow. Thus if the arrow weighed 500 grains, the bow would be defined to have a virtual mass of 250 grains.

Oneida Labs publishes a virtual mass of 176.3 grains for their Oneida Eagle bow when set for a peak draw of 60#. Thus, the Oneida Eagle bow when shooting an arrow three times as heavy, i.e. 176.3 x 3 = 528.9 grains, should have an efficiency of 3/4ths or 75%, which it does.

The concept of "virtual mass" would be useless if the bow's virtual mass computed to be a different number under different circumstances. According to Paul E. Klopsteg's article "Physics of Bow and Arrows" published in the Aug. 1943 issue of American Journal of Physics, the virtual mass of a bow is in fact a constant. He tested a large number of bows shooting vastly different weight arrows and satisfied himself that each bow had a constant virtual mass. In his day, of course, the adjustable-draw-weight bow had not yet been introduced. Oneida's published data on their Eagle bow shows slightly different virtual masses for the same bow set at different weights. Their data is:

Peak draw force	50#	55#	60#
Virtual mass	171	169	176 gr.

It is not clear from inspection of these numbers if the differing virtual masses are real or are simply a result of measurement dispersions.

Over-draw Conversion:

The usefulness of the virtual mass concept can be illustrated by recounting the author's experience in converting his bow from standard to "over-draw". An "over-draw" conversion is the addition of an arrow rest about three inches back from the conventional location. The reason to do this is to be able to shoot shorter, lighter, and thus faster, arrows. Shorter arrows weigh less not only because they are shorter but also (and mainly) because they can have thinner walls. I wanted this conversion because I was totally convinced of the usefulness of speed in offsetting errors in the estimation of target range. The archery shop owner assured me that I would get at least a 15% speed increase. I took his word for it, but he was wrong. As a mechanical engineer, I should have known on two counts.

First, knowing nothing about the "virtual mass" concept, I still should have known that the amount of energy imparted to the arrow would be unchanged, at best. Knowing that, it would have been easy to compute the speed change based simply on the arrow weight reduction. The original arrows were 29" long, 21/64" diameter, 0.017" thick wall arrows commonly known as "2117"; and they weighed 545 grains. The over-draw arrows were 26" long, 21/64" diameter, 0.014" thick wall arrows commonly known as "2114"; and they weighed 460 grains. The ratio of 545/460 is 1.185. Since energy of an arrow is proportional to the velocity squared, the maximum velocity ratio would be the square root of 1.185, which is 1.088. Thus I should not have hoped for more than an 8.8% speed increase. The "virtual mass" concept, however, would have predicted a speed increase of only 6.0%. The underlying reason is that with a faster arrow, the parts of bow are also moving faster and thus retaining more energy. The calculation for my bow, before and after, is as follows:

Data:

Input energy = 69.2 foot-pounds.
Hysteresis = (about) 8% of input = 0.08 x 69.2ft-lb = 5.5 ft-lb.
Velocity of standard arrow = 193 feet/second (for 29" 2117 arrow).
Weight of standard arrow = 545 grains (for 29" 2117 arrow).
Weight of overdraw arrow = 460 grains (for 26" 2114 arrow).

Steps in Calculation of bow's virtual mass:

First: Calculate arrrow's energy.

$$\text{Arrow's energy} = mV^2/2g$$

$$\text{Arrow's energy} = (545\text{gr}/7000\text{gr}/\#) \times (193\ \text{ft/sec})^2 / 2 \times 32.2\ \text{ft/sec}^2.$$

$$\text{Arrow's energy} = 0.0779\# \times 37{,}249\ \text{ft} / 64.4 = 45.0\ \text{ft-}\#.$$

Second: Calculate missing energy.

Missing energy = input - hysteresis - arrow's = 69.2 - 5.5 - 45.0 = 18.7 ft-#.

(continued next page)

Third: Calculate virtual mass.

There are several ways to do this. The easier is to use the ratio of missing energy to arrow energy, like so:

$$\frac{\text{Virtual mass}}{\text{Arrow weight}} = \frac{\text{missing energy}}{\text{arrow energy}}$$

$$= \frac{18.7 \text{ ft-\#}}{45.0 \text{ ft-\#}} = 0.416$$

Virtual mass = 0.416 x 545 grains
= 226 grains.

Calculation of Predicted Velocity:

The output energy and the bow's virtual mass will remain unchanged after the "overdraw" conversion. Thus, new data shall become:

E,out = 63.7 ft-lbs.

VM,bow = 226 grains.

W,arrow = 460 grains.

$E,out = (W,arrow + VM,bow) \times V^2/2g$

$V^2 = E,out \times 2g/(W,arrow + VM,bow)$

$$V^2 = \frac{63.7\text{ft \# } \times 2 \times 32.2 \text{ ft/sec}^2}{(460 + 226)\text{gr}/7000 \text{ gr/\#}}$$

$$= 41{,}860 \text{ ft}^2/\text{sec}^2$$

V = 205 ft/sec. = predicted velocity for lighter arrow.

Notice that the velocity predicted using the virtual mass concept is 6.0% faster (205fps/193fps = 1.06). The velocity predicted using the constant energy (to the arrow) concept was 8.8%. As best I was able to determine, the actual velocity increase was a little less than 6%.

"Dry-fire Velocity:

A bow's efficiency is zero when fired with no arrow. The virtual mass concept can let you predict the highest speed achievable with an arrow weighing nothing at all. For my own bow, using the data above:

$$V^2 \times (VM,bow)/2g = E,out$$

$$V^2 = 2g \times E,out/VM,bow$$

$$= \frac{2 \times 32.2\text{ft/sec}^2 \times 63.7\text{ft-\#}}{(226 \text{ gr}/ 7000 \text{ gr/\#})}$$

$$= 127{,}062 \text{ ft}^2/\text{sec}^2$$

V, max = 356 fps.

Crowbar Velocity:

Visualize shooting an ultra-heavy arrow in the form of a 50# crowbar mounted on roller skates. The weight of the crowbar would be so great when compared to the virtual mass of the bow that virtually all of the bow's output energy would go to the crowbar. This is another way of saying that the launch velocity would be so slow that the kinetic energy remaining in the bow parts would be negligible. Using the author's bow as an example:

$V^2 \times (W,crowbar)/2g = E,out$

$V^2 = 2g \times E,out/W,crowbar$

$= 2 \times 32.2 \text{ ft/sec}^2 \times 63.7\text{ft-\#}/50\#$

$V^2 = 82 \text{ ft}^2/\text{sec}^2$

V,crowbar = 9.1 fps = 6.2 mph.

Interrelationships between Arrow Weight, Virtual Mass & Efficiency:

A bow's "virtual mass" is a much more meaningful number than is it's efficiency. Efficiency varies with the weight of the arrow. All bows have zero efficiency when shooting zero-weight arrows. All bows are as efficient as limited by their hysteresis when shooting ultra-heavy arrows. A bow's "virtual mass" can be used directly to ascertain the fraction of energy which will be imparted to the arrow. The following table assumes a bow has a virtual mass of 200 grains, and shows the ratios of energy imparted to arrows of different weights.

Virtual mass (gr)	Arrow weight (gr)	Total (gr)	Arrow's Energy (%)	Arrow's Speed
200	0	200	0%	200% = Dry fire.
200	200	400	50%	141%
200	400	600	67%	115% = Light weight arrow.
200	500	700	71%	107% = 2016 x 30" broadhead.
200	**600**	**800**	**75%**	**100%** = 2219 x 30" broadhead.
200	800	1,000	80%	89%
200	1,800	2,000	90%	63%
200	6,750	6,950	97%	34% = Steel 3/8" x 30" rod.
200	22,310	22,510	99%	19% = Steel 11/16" x 30" rod.

Table 3-1 ... Energy Absorbed & Speed Achieved versus Arrow Weight

Table 3-1 was calculated as follows:

First, a "reference" situation of a 600 grain arrow matched with a bow having a virtual mass of 200 grains was taken as a starting point having, by definition, 100% of reference speed.

Second, the arrow's energy is computed as equal to the weight of the arrow divided by the sum of the weight of the arrow plus the weight of the bow's virtual mass. For instance, for the reference point, arrow energy is 600gr/800gr = 75%.

Third, the arrow's speed was computed by looking at the comparison of total weights and taking the square root thereof. For example, in the "dry fire" situation the ratio of total weights is 800 grains (for the reference) to 200 grains for the dry fired bow. The ratio is 8/2 = 4. The square root of 4 is 2, which is 200%.

Variation of Virtual Mass

Back in 1940 or so when Paul E. Klopsteg came up with the "virtual mass" concept, he said that a bow's virtual mass is a constant. I think that most engineer/archers had to know that there was no physical explanation why such should be exactly true. None-the-less, since it was obviously an excellent mathematical tool closely approximating actual field measurements, there was no reason to reject the concept. In 1987, Norbert F. Mullaney, P.E., sent me a copy of his "white paper" on the topic of virtual mass wherein he explained that a bow's virtual mass is not in fact constant. Reviewing his data, I formulated the following phrase to describe his findings:

"A bow's virtual mass is a constant plus a small percentage of the arrow's weight."

The small percentages were shown in Norb's formulas to be:

Jennings Split T	6.9%
Bear Custom Kodiak T/D	13.2%
Stewart 62" Recurve	0.1%
Browning Fire-Drake Recurve	10.5%
(unidentified bow)	5.9%

In the Auguest 1990 issue of Bowhunting World (published in May 1990), Norb Mullaney, P.E., came up with new information on the variation of virtual mass. Buried in his special longbow report, "A Longbow Called Grande" he showed how the virtual mass of different type bows vary with changes of arrow weight.

Translating his data into the same format as above, the percentages of arrows' weight by which virtual masses changed were:

Recurved limb compound	+9.9%
Straight limb compound	+2.6%
62" recurve	+0.6%
68" Grande longbow	-6.8%

Conclusion

Bows are fairly efficient machines. They are impractical when used at the upper or at the lower limits of efficiencies. Thus the trade-offs faced by bowmakers and by bow users will always leave a lot of room for differences of opinion.

*** END ***

The conversion of work stored within the bow to kinetic energy of an arrow is quite efficient. Order-of-magnitude numbers accounting for the energy which the archer puts into the bow upon drawing are:

Arrow propulsion:	69%
Bow's "virtual mass":	23%
Bow's hysteresis:	8%.
Arrow's vibration & rotations:	Negligible

See Chapter 1 for a discussion of the work the archer does upon drawing. See Chapter 3 for a discussion of "virtual mass", which is a concept devised to put a number on the amount of kinetic energy left behind in the bow at the moment the arrow leaves. See Chapter 2 for a discussion of the bow's hysteresis, which is the energy lost due to internal friction within the bow.

The energy imparted to an arrow in the form of forward kinetic energy dwarfs other forms of energy imparted to the same arrow, such as rotational and vibrational energy.

Velocity energy (or propulsion energy) of an arrow is its mass multiplied by the square of its velocity:

$E = W \times V^2/2g$ where

W = weight of arrow

V = velocity

g = gravity = 32.2 ft/sec^2

Example:

An arrow weighing 546 grains is shot at 180 ft/sec. What is it's energy? Answer:

$$E = \frac{546\text{gr } (180 \text{ ft/sec})^2}{2 \times 32.2\text{ft/sec}^2 \times 7000\text{gr/\#}}$$

E = 39.2 foot-pounds.

The opposite calculation can be made. Given the energy expected to be imparted to the arrow, how fast will it go? Answer:

$$E = W \times V^2/2g.$$

Solving for V^2 yields:

$$V^2 = 2g \times E/W$$

Example:

Arrow weighs 546 grains. 45 foot-pounds are predicted as arrow propulsion energy. What will arrow speed be?

$$V^2 = 2g \times E/W$$

$$V^2 = \frac{2 \times 32.2\text{ft/sec}^2 \times 45 \text{ ft-\#}}{546 \text{ gr} / 7000 \text{ gr/\#}}$$

$$V^2 = 37{,}154 \text{ ft}^2/\text{sec}^2$$

$$V = 193 \text{ ft/sec.}$$

Elevation Energy

There is a conversion between speed and altitude given by the formula:

$$A = V^2/2g.$$

This will be discussed in more detail in the chapter on trajectories. When hunting, it is important to realize that when shooting uphill the arrow will arrive traveling slower and will thus have reduced penetration power.

Energy of Rotation & of Vibration:

Vibration, rotation, fish-tailing and porpoising are forms of kinetic energy. Their magnitudes are calculated in the following paragraphs and are shown to be insignificant when compared to the arrow's total energy. The calculations do not shed any light upon how important such energy forms are to accuracy.

Rotational energy is zero while the arrow is being launched, because the nock prevents the arrow from rotating. As soon as the arrow clears the string, it is free to rotate as dictated by the fletching's spiral. I measured the rotation of my 5" plastic 3-fletch factory arrows and found that they made one revolution while advancing 5.1 feet. I did this by taking the arrows under water with me in the swimming pool and counting the revolutions made while they floated from the bottom to the surface. Knowing the arrow's forward speed makes it possible to compute the revolutions per second.

Example:

Arrow's forward speed = 180fps.
Rotation = 1.0 revolution per 5.1 feet advance.
Calculate revolutions per second.

RPS = 180 ft/sec x 1 rev/5.1ft
RPS = 35.3 rev/sec

The rotational energy is similar to the forward energy. It is mass x velocity squared, but in this case the velocity is circular. For the analysis of an arrow, it is easiest to ignore the fact that the tip and fletching have different distances from the arrow's center than do the walls of the arrow. To figure out the velocity of the arrow's walls, use their average radius.

Example:

Arrow is "2117", having outside diameter = 21/64".
Arrow's wall thickness = 17/1000ths.
Average radius
= 1/2 x (21/64"- 0.017")
= 0.1556"
Velocity
= pi x 2 x radius x rps
= 3.1415 x 2 x 0.1556" x 35rps
= 34.5"/sec/12"/ft = 2.85 fps

$$\text{Energy} = \frac{546 \text{ gr } (2.85\text{ft/sec})^2}{2 \times 32.2\text{ft/sec}^2}$$

= 68.9 ft-grains /7000 gr/#
= 0.10 ft-pounds.

Note that this amounts to less than one tenth of one percent of the energy of forward motion.

Conclusion: Rotational energy is negligible. Note that this conclusion is different than for bullets. Bullets, however, depend upon the gyroscopic feature of high rotational speed to avoid tumbling. Arrows rely upon their fletching.

The reason for an arrow to be rotated at all is to average out the up-down and left-right aerodynamic profile presented by the fletching and by the broadhead.

Fish-tailing &/or Porpoising Energy:

The "fish-tailing" and/or "porpoising" of an arrow represents rotational energy, with rotation being about a vertical axis. A badly fish-tailing arrow looks to the eye as though it would (were it not for fletching) complete one revolution during a 40-yard shot. The amount of energy this represents is:

E = weight x pi x 2 x (avg. radius) x rotational speed squared / 2 x g

Example:

Assume arrow is 546 grains, 2117, 28" long, with weight (on average) located 10" out from fore-and-aft center of arrow, fired at 180 ft/sec.

Rotational speed is one revolution in (40 x 3 feet)/(180 ft/sec) = 1.5 rev/sec. Rotational velocity is 1.5 rev/sec x 10/12 ft x 3.1416 x 2 = 7.85 ft/sec.

$$E = \frac{546\ gr\ (7.85\ ft/sec)^2}{2 \times 32.2 ft/sec^2 \times 7000 gr/\#}$$

E = 0.07 foot-pounds.

Note that this is insignificant compared to forward speed's energy.

Vibrational Energy

A pole-vaulter uses the energy stored within his (bent) pole to hoist him up and over. An arrow is similarly bent while being accelerated. Just how that energy is accounted for and what influence it has on performance is unknown to this author. It may very well be quite significant, and it may well be inter-related with consideration of arrow spine. On one hand, the vibrations may cause the arrow's tail to wag, resulting in a parachute type slowing of the arrow. On the other hand, it may somehow propel the arrow in a manner similar to a Chinese boatman sculling. The amplitude of this vibrational energy must be negligible compared to that of forward velocity because the hysteresis and virtual mass concepts of energy accounting work well while ignoring vibration. The author mentions these considerations without shedding any light simply in the interest of attempting to give a complete accounting of the various forms of energy.

Heat Energy

Eventually, all of an arrow's energy ends up in the form of heat. The arrow heats the air though which it passes just a little bit. When it hits the target, both the target and the arrow are heated by the resulting friction. Just as a matter of passing interest, the amount by which an arrow will be warmed is calculated. For simplicity, it will be assumed that the arrow arrives with 100% of the energy it had at launch. Further, it will be assumed that all of this energy ends up heating the arrow, not the target. A final simplifying assumption is that the arrow is 100% aluminum.

Data for author's 69# compound with overdraw conversion:

Energy = 63.7 foot-pounds.
Weight = 460 grains .
Specific heat of aluminum = 0.22 Btu/pound°F.
One pound = 7,000 grains
One BTU = 778 ft-lbs.

$$\text{Temp rise} = \frac{\text{energy}}{\text{weight x specific heat}}$$

$$\text{Temp rise} = \frac{63.7\ ft\text{-}\#}{460\ gr \times 0.22\ Btu/\#F°}$$

$$\text{Temp rise} = \frac{0.63 \frac{ft.\#\ F°}{gr\ BTU} \times 7000 \frac{gr}{\#}}{778\ ft.\#/BTU}$$

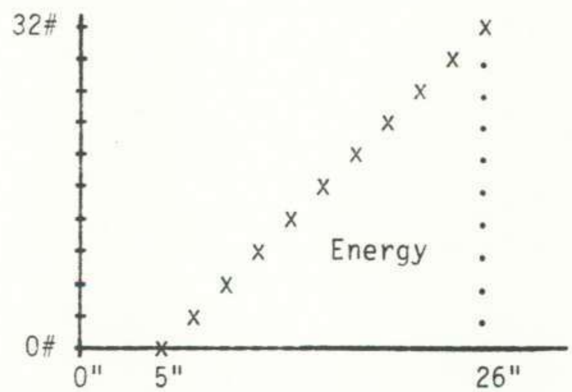
= 5.7 F°.

Next time you withdraw your arrow, see if you cannot notice it's warmth!

*** END ***

The mathematical description of an arrow's acceleration, velocity and location at each moment between the time the string is released and the time the arrow leaves the string will be computed. A compound bow's non-linear force-draw curve makes it difficult to handle mathematically. The recurve &/or longbow, however, have a nearly linear force-draw curve. Thus the recurve &/or longbow can more easily be mathematically analyzed.

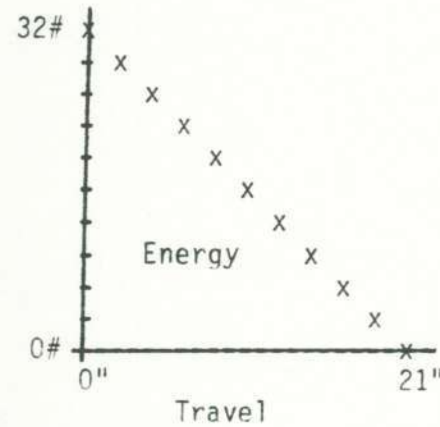
The force-draw curve for a 32# fiber-backed lemonwood pulling 0# @ 5" and 32# at 26" looks as follows:



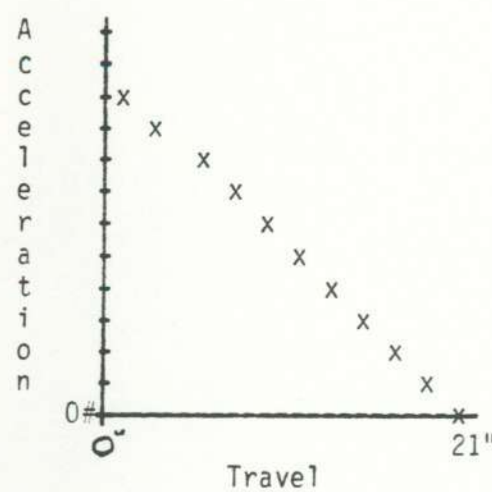
The area under the curve represents stored energy.

Acceleration

The forces delivered to the arrow (and to those parts of the bow which are also accelerated) are the reverse of the force of drawing the bow. Re-drawing the bow's force-draw curve, but re-orienting it to visualize the arrow being shot from left to right yields:



Acceleration equals force divided by mass, i.e.: $a = F/m$. Since force, as shown above, is a lineal function of distance, so too is acceleration. Thus acceleration versus travel from point string leaves the archer's fingers to the point where arrow and string part company is as shown in the following diagram.



Peak acceleration is the initial acceleration. The mass to be accelerated includes the arrow plus those parts of the bow which are its "virtual mass".

Virtual mass

Virtual mass is that mass which, were it going at the same velocity as the arrow, would account for all the energy for which the arrow does not account. In our example, ignoring the 3% to 8% hysteresis loss, the area under the force-draw curve is:

$\frac{1}{2} \times 32\# \times 21'' / 12''/\# = 28.0$ ft-lb.

This bow imparts a velocity of 114 ft/sect to an arrow weighing 545 grains. The kinetic energy in the arrow at the moment of launch is:

$$E_{,arrow} = 1/2\ m\ V^2$$

$$E_{,arrow} = \frac{545\,gr\left(114\,\frac{ft}{sec}\right)^2}{2 \times 7000\,\frac{gr}{\#} \times 32.2\,\frac{ft}{sec^2}}$$

$$E_{,arrow} = 15.71 \text{ ft-lbs.}$$

The missing energy is considered to be in the bow's "virtual mass". The missing energy is:

$$E,bow = E,input - E,arrow$$

$$= 28.00 - 15.71 = 12.29 \text{ ft-lbs.}$$

The virtual mass needed to account for the missing energy is:

$$VM = W,arrow \times E,bow/E,arrow$$

$$= 545 \text{ grains} \times 12.29/15.71$$

$$VM = 426.3 \text{ grains.}$$

The total mass to be accelerated is that of the arrow (545 grains) plus the bow's virtual mass (426.3 grains), or 971 grains total.

Peak Acceleration:

Now that the total mass has been computed, the peak (&/or initial) acceleration can be computed:

$$a,max = F/(W,arrow + V.M.)$$

$$= 32\#/(545 + 426.3)\text{grains}$$

$$= \frac{32\# \times 7000\,\frac{gr}{\#} \times 32.2\,\frac{ft}{sec^2}}{971.3\,gr}$$

$$= 7{,}426 \text{ ft/sec}^2.$$

Dividing by 32.2 ft/sec^2 yields the "g"-force, which is a tremendous 231-g's!

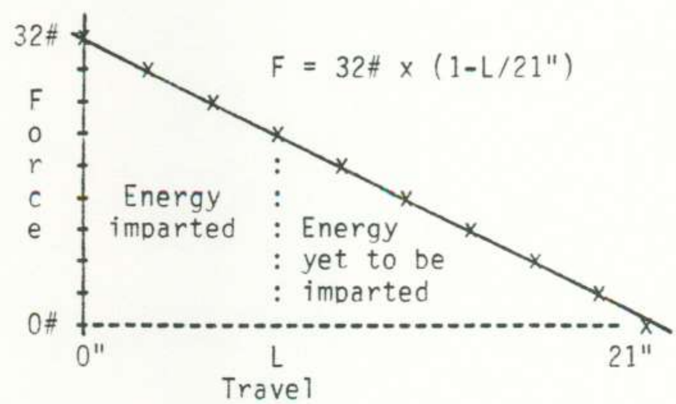
Velocity

The variation of velocity with travel can most easily be found by noting that we know how much energy remains under the force-draw curve for any given point. Thus we can compute how much energy has been delivered to the arrow and virtual mass for any given point. Knowing the arrow's energy, it is an easy task to compute it's speed using the formula:

$$E = 1/2 \times m \times V^2$$

$$= (W,arrow + V.M.) \times V^2/2g.$$

To derive a formula for energy at any point, the force-draw curve is re-drawn showing an arbitrary length of travel, "L":



The force at point "L" is:

$$F = 32\# \times (1 - L/21'')$$
$$F = F,max(1 - L/L,max).$$

The average force between starting point and point "L" is:

$$F_{avg} = 1/2 \times (F_{max} + F_{L})$$

$$= \frac{1}{2}\left(F_{max} + F_{max}\left(1 - \frac{L}{L_{MAX}}\right)\right)$$

$$= \frac{F_{max}}{2} \times (2 - L/L_{max}).$$

The energy imparted is the average force multiplied by the travel:

$$E_{imparted} = F_{avg} \times L$$

$$E_{imparted} = \frac{F_{max}}{2}\left(2 - \frac{L}{L_{max}}\right)L$$

Conversion of energy using

$E = mV^2/2g$ yields:

$$V^2 = 2g \times E/(W_{arrow} + VM)$$

$$V^2 = \frac{2g}{(W+VM)} \cdot E$$

$$V^2 = \frac{2g}{(W+VM)} \cdot \frac{F_{max}}{2}\left(2 - \frac{L}{L_{max}}\right)L$$

$$= \frac{g \times F_{max} \times (2 - L/L_{max}) \times L}{W + VM}$$

Example:

For our sample bow:

$$W + VM = \frac{545 gr + 426.3 gr}{7000 gr/\#} = 0.139\#$$

$L_{max} = 21''/12''/ft = 1.75$ ft

$F_{max} = 32\#.$

$$V^2 = \frac{32.2\frac{ft}{sec^2} \times 32\#\left(2 - \frac{L}{1.75ft}\right) \times L}{0.139\#}$$

$$= 7426\ ft^2/sec^2\left(2 - \frac{L}{1.75ft}\right)$$

A resulting table of values is:

Travel (inches)	Velocity (ft/sec)
.0	.0
.5	24.7
1.0	34.8
2.0	48.6
4.0	66.9
6.0	79.8
8.0	89.5
10.0	97.1
12.0	103.0
14.0	107.5
16.0	110.7
18.0	112.8
20.0	113.9
21.0	114.0

Time versus Speed &/or Acceleration:

The preceeding analysis described accelerations and velocities of the arrow versus positions of the arrow as it traveled from the finger release point to to the point of separation of string and arrow. The analysis was totally mute on when these events would take place. Now we will work up the formulas needed to describe acceleration, velocity and position for any given moment following release of the arrow up to the moment when the arrow separates from the string.

Referring back to my undergraduate text book on dynamics ("Mechanics, Part II: Dynamics" by J. L. Meriam of the University of California, Berkeley) the math which applies is that which describes a slider block attached to a spring. On Pages 22, 23 & 24, Professor Meriam works up the following formulas:

$a = -k^2 s$ where a = acceleration

k = a constant

s = displacement

$v = V_o \cos kt$ v = velocity

v_o = velocity, original

t = time

$s = (v_o/k)\sin kt$

$T = 2\pi/k$ T = period of cycle

$s = A \sin kt + B \cos kt$

$v = Ak \cos kt - Bk \sin kt$

Translating all the above to what described an arrow launch from a recurve bow was a bit challenging, but I finally worked up the following:

$-k^2$ = bow's spring constant divided by mass of arrow and bow's virtual mass.

$-k^2 = Cg/(W_{,arrow} + W_{,bow})$.

C = peak draw weight divided by draw length.

$C = F_{,max}/L_{,max}$.

$a = -kV_{,max} \sin kt$.

$V = V_{,max} \cos kt$.

$L = (V_{,max}/k) \sin kt$.

The trick to using the above formulas is to realize that "kt" is in radians. For most calculators, it is necessary to convert radians into degrees. Further, the only applicable values of "kt" are those between 0° and 90°. Thus the first thing about launch time to be calculated is the total duration of the launch, which occurs when "kt" equals 90°. Ninety degrees equals 90 x pi/180 = 1.571 radians. Thus the duration of a launch is:

$kt_{,max} = 1.571$

$t_{,max} = 1.571/k$

The next items to get straight when using the above formulas is whether t = 0 at the moment of string release or at the moment of arrow separation. Finally, it is necessary to keep straight if L = 0 at the moment of string release or at the moment of arrow separation. I suggest not worrying about which is which, but if the answer comes out backwards, simply reverse it.

Example:

Using the sample bow which developes 32# pull at 21" of draw and which fires a 545 grain arrow at 114 ft/sec and which has a virtual mass of 426.3 grains:

C = 32#/21" = 32# x 12"/ft / 21"

= 18.286#/ft

$W_{,arrow} + W_{,bow}$

= (545+426.3)grains/7000gr/#

$W_{,arrow} + W_{,bow} = 0.1388\#$.

$k = -(Cg/(W_{,arrow} + W_{,bow}))^{1/2}$

$$k = -\left[\frac{18.286\frac{\#}{ft} \times 32.2\frac{ft}{sec^2}}{0.1388\#}\right]^{\frac{1}{2}}$$

$k = -(4{,}243/sec^2)^{1/2}$

$k = -65.14/sec$

$t_{,max} = 1.571/k = 1.571/(-65.14/sec)$

$t_{,max} = -0.0241$ seconds = total period of launch.

Repeating the formulas for each of the parameters:

$a = -kV_{,max} \sin kt$

$$a = \frac{65.14}{sec} \times 114\frac{ft}{sec} \cdot \sin\left(\frac{-65.14\,t}{sec}\right)$$

$$a = 7{,}426 \frac{ft}{sec^2} \sin\left(-\frac{65.14 t}{sec}\right)$$

$$V = V_{max} \cos kt$$

$$V = 114 \text{ ft/sec} \cos(-65.14/\text{sec} \times t).$$

$$L = (V_{max}/k) \sin kt$$

$$L = \left(\frac{114 \text{ ft/sec}}{-65.14/\text{sec}}\right) \sin\left(\frac{-65.14\, t}{sec}\right)$$

$$L = 1.75 \text{ ft} \sin(-65.14/\text{sec} \times t).$$

Solving and putting in table form yields the following:

Table of acceleration, velocity & travel versus time

t (sec)	kt (rad)	Kt (deg)	sin kt	a (fps^2)	cos kt	V (fps)	L (ft)	L (in)	t (sec) Rearranged
-.0241	-1.571	-90.0	-1.000	-7,426	.000	.0	-1.75	.0	.0000
-.0214	-1.396	-80.0	-.985	-7,313	.174	19.8	-1.72	.3	.0027
-.0187	-1.222	-70.0	-.940	-6,978	.342	39.0	-1.64	1.3	.0054
-.0161	-1.047	-60.0	-.866	-6,431	.500	57.0	-1.52	2.8	.0080
-.0134	-.873	-50.0	-.766	-5,689	.643	73.3	-1.34	4.9	.0107
-.0107	-.698	-40.0	-.643	-4,773	.766	87.3	-1.12	7.5	.0134
-.0080	-.524	-30.0	-.500	-3,713	.866	98.7	-.875	10.5	.0161
-.0054	-.349	-20.0	-.342	-2,540	.940	107.1	-.599	13.8	.0187
-.0027	-.175	-10.0	-.174	-1,290	.985	112.3	-.304	17.4	.0214
0	.000	.0	.000	0	1.000	114.0	0	21.0	.0241

CHART ANALYSIS:

The above data is plotted on Figure 5-1. Acceleration and velocity are also plotted versus travel in the same figure. A study of those graphs leads me to the following conclusions:

Velocity versus travel shows that half of peak velocity is achieved in the first 3" of travel.

Velocity versus time shows that velocity increases almost uniformly with time for about the first 2/3rds of the launch period.

Travel versus time shows that the arrow hardly moves at all during the first 25% of the launch period.

Travel versus time shows that half of the arrow's movement occurs during the last 15% of the launch period.

Acceleration versus time shows that peak acceleration prevails for a relatively long time.

The time graph shows that during the first half of the launch period, the arrow accelerates at a tremendous rate but does not go very far. During the second half, acceleration drops to nothing but the arrow moves a great deal.

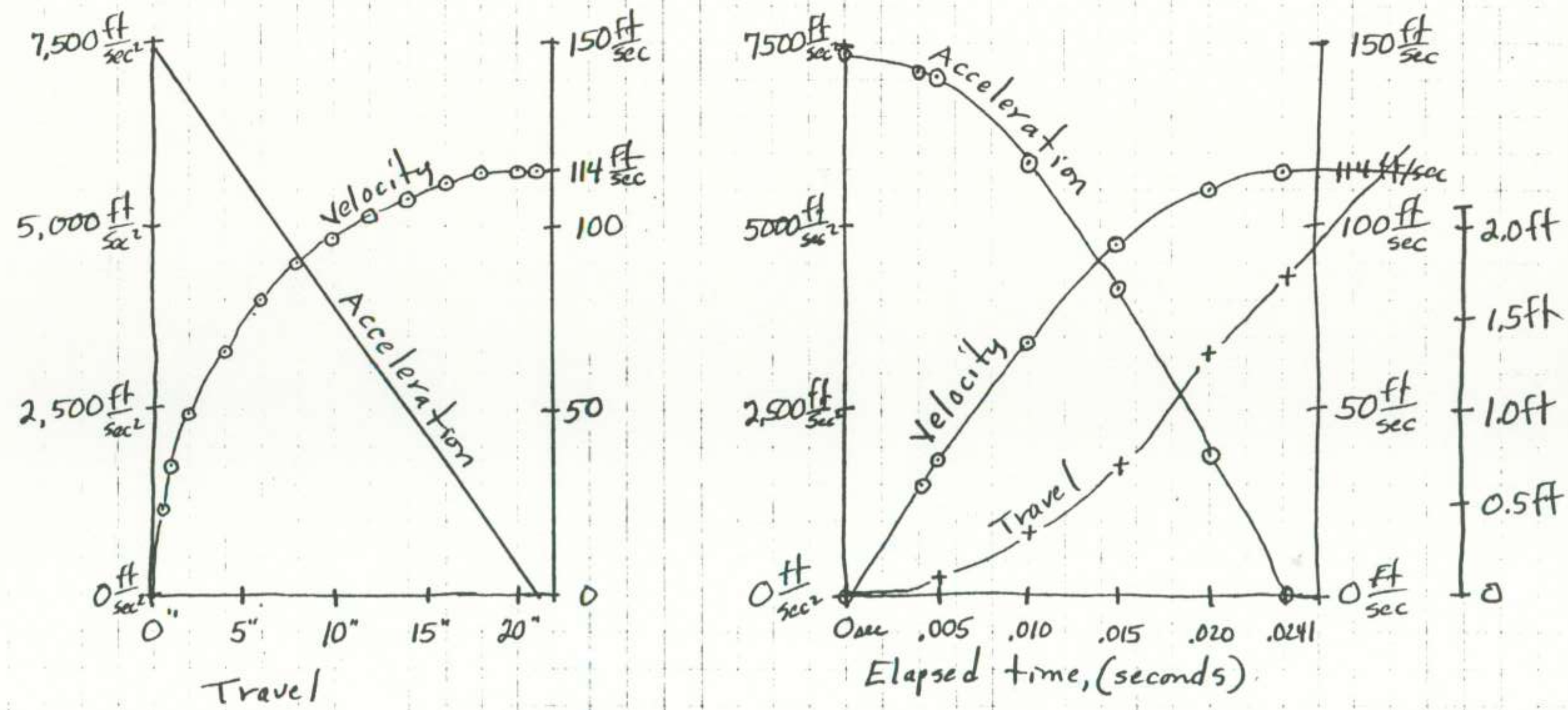


Figure 5-1 Launch Accelerations, Velocities + Travel

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June 30, 1986

Mr. Thomas L. Liston P.E.
Liston and Associates, Inc.
890 Saratoga Avenue
San Jose, CA 95129

Dear Thomas:

In reviewing your paper "Firing Kinetics", I feel that your mathematical approach may well be a valid statement for perfect conditions of release and initial launch of an arrow, but it fails to consider conditions that exist in the "real life" event.

In the first place, at the instant of release, the bow string does not immediately apply the holding force to the arrow. The bow string, whether it is held by the bowman's fingers or with the use of a release aid, has a discontinuity in the area of the nocking point. When the string is released and the limbs start their return to brace height position, this discontinuity must be eliminated before the force of the string can be fully applied to the arrow. This action causes a delay in the application of the full string force and hence the instantaneous acceleration of the arrow.

Over and above this string discontinuity, we also have a buckling action of the arrow shaft. This buckling action is magnified with the finger release but it is present to some degree even with the use of sophisticated release aids. I believe that study would show that the bow string itself, under the influence of discontinuity elimination and the forces of acceleration, acts somewhat elastically further softening the initial application of the string force on the arrow.

Dr. Dallas Smith of Tennessee Technological University studied arrow launch with strobe photography. He used a compound bow for part of the study and a recurved bow for the remainder. Smith was able to measure displacement at an initial time lapse of just over 5 milliseconds and thereafter at increments of about 2.5 milliseconds.

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His plots of propulsion force (string force) applied to the arrow reflect the difference that exists between the draw force characteristics of the recurve and the compound. He extrapolates the acceleration curve of the compound to a zero intercept of about 3350 feet per second/second but the extrapolation covers about 8 milliseconds after release.

He did not present similar data for the recurved bow he tested but it is logical to assume that he would have projected a greater zero intercept.

It is my personal opinion that the initial acceleration is not zero but has some finite value that is substantially lower than pure theory would envision. I believe that you are correct in assuming that Hickman erred in the shape of his acceleration curve, but Smith writes of having used the laws of impulse-momentum and work-kinetic energy to aid in the extrapolation. He makes no mention of the initial softening of the applied propulsion force for the reasons I cited.

I have not personally conducted any experimental analysis of the internal ballistics of the bow and arrow combination. My observations and analysis are based on the work of others, but I cannot rationalize instantaneous high level acceleration considering the events that I know to occur.

Sincerely,

Norb Mullaney

Norb Mullaney

Frictionless Flight

Arrows flying short distances (less than 100 yards) with streamlined fletching (as opposed to "flu-flu" fletching) behave somewhat as though there were no significant friction. See the chapter, "Trajectories with Friction." In this chapter, only frictionless flight is considered. The trajectory of an arrow in flight is much the same as that of any other projectile, such as an artillery shell, a thrown rock or the stream of water from a garden hose. The two beginning formulas from which all others are derived are:

$$x = tV \cos A \qquad \text{Equation \#6-1}$$

$$y = tV \sin A - \tfrac{1}{2}gt^2 \qquad \text{Eqn \#6-2}$$

where:
t = time elapsed since launch.
x = horizontal distance.
y = elevation compared to launch.
A = angle of launch.
V = velocity at moment of launch.
g = gravity = 32.2 ft/sec^2.

Equation #6-1 says that the horizontal component of velocity is constant throughout the arrow's flight. Consequently, the distance traveled in the horizontal direction is a direct function of elapsed time.

Equation #6-2 utilizes the idea that the arrow's height depends upon it's initial upward component of velocity, which is continuously changed by gravity. The formula holds for negative values of height, such as when firing an arrow off a cliff into a valley.

Elimination of time between Equations #6-1 & #6-2 yields an equation for the trajectory:

Equation #6-3 ...

$$y = x \tan A - gx^2/(2V^2\cos^2 A)$$

Each of the preceding three formulas are valid for arrows shot in any direction, including down.

Limit of all possible trajectories:

An equation to calculate the envelope of all possible trajectories of projectiles fired at the same velocity is:

Equation #6-7 ...

$$y = (V^2/2g) - (gx^2)/(2V^2)$$

Figure 6-1 shows two typical trajectories plus an envelope of all possible trajectories, and Fig. 6-2 is the same except it is to-scale for a launch velocity of 200 ft per second.

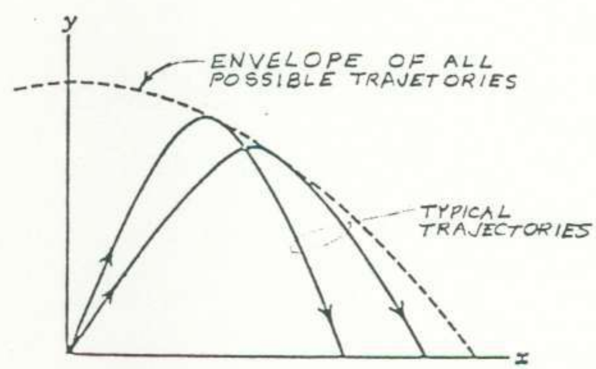


Fig. 6-1 ... TRAJECTORIES

Maximum horizontal range:

Maximum horizontal range of projectile fired at 45° above horizontal is:

Equation #6-4

$$R_{max} = V^2/g$$

where R_{max} = maximum range.

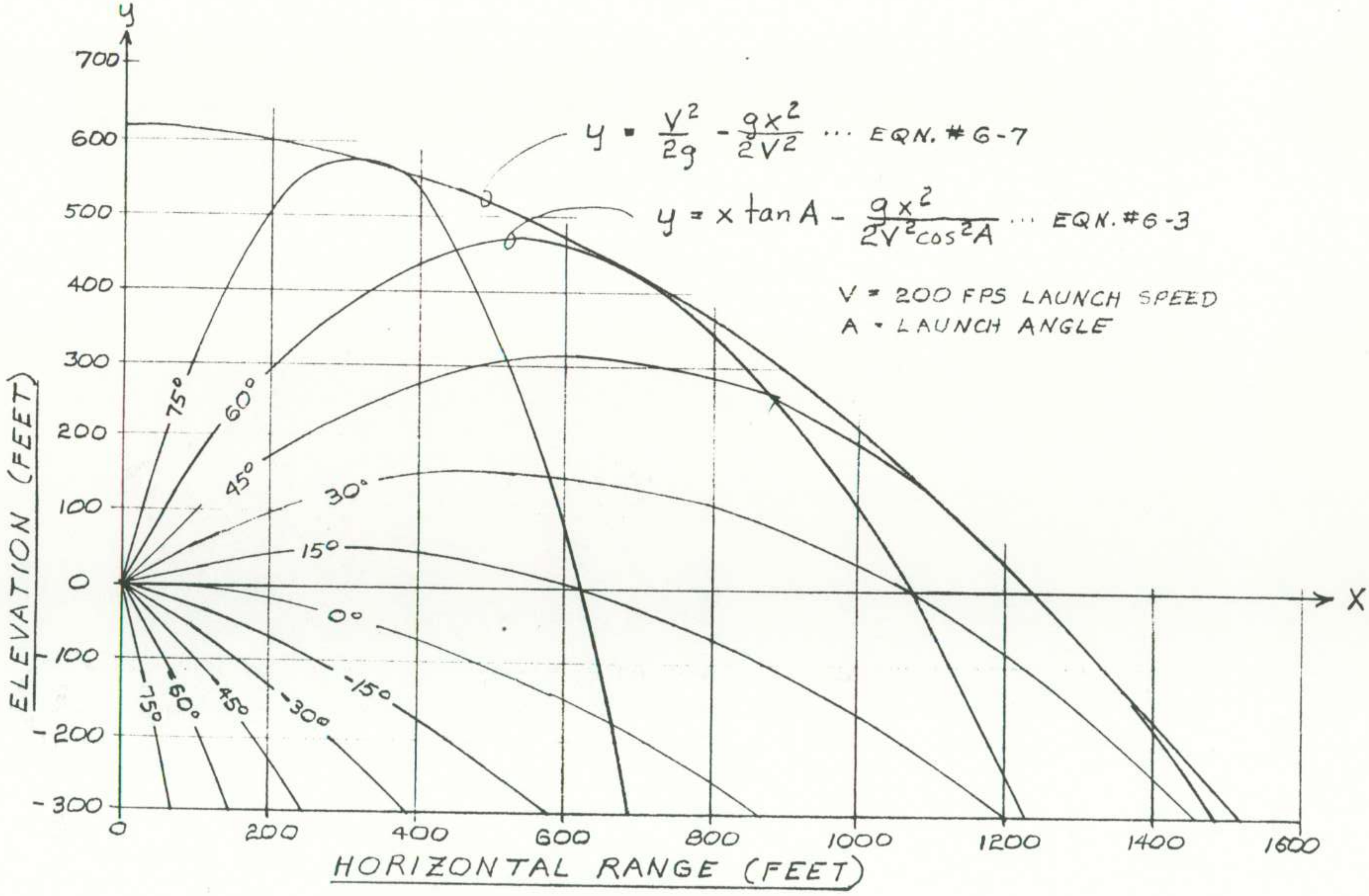


Figure 6-2 ... Frictionless Trajectories

Time of flight:

The time of flight of a projectile fired at a target having the same elevation as the launch point:

Equation #6-5

$$t = (2V \text{ sine } A)/g$$

Arrow's speed:

By shooting an arrow straight up in a frictionless environment (such as on the moon) and noting the time from launch to return, firing velocity can be computed as:

Equation #6-6 $$V = tg/2$$

Apogee:

The zenith altitude, h, is:

Equation #6-8 $$h = V^2 \sin^2 A/(2g)$$

Maximum altitude:

The maximum altitude possible when shooting straight up is:

Equation #6-9 $$h_{,max} = V^2/2g$$

Range:

The horizontal distance an arrow will travel when fired at an angle A above the horizontal with an initial velocity V at a target having the same elevation as the launch elevation is:

Equation #6-10 $$R = (V^2/g) \sin(2A)$$

Elevation angle:

The elevation angle needed in order to fire an arrow at a horizontal range R is:

Eqn. #6-11 $$A = \tfrac{1}{2} \arcsin (Rg/V^2)$$

where "arcsin" means "angle whose sine is".

Change of range versus change in elevation:

Change of range versus change in arrow elevation for a horizontal shot is given by:

Eqn. #6-12 $$dR/dA = (-2V^2/g) \cos (2A)$$

All ordinary archery shots use elevations of 0 to 8°. At higher angles, the target is obscured by the arrow. Therefore,

$$\begin{aligned} \cos (2A) &= \cos (0° \text{ to } 2 \times 8°) \\ &= \cos (0° \text{ to } 16°) \\ &= 1.000 \text{ to } 0.961. \end{aligned}$$

The change of range for a change of elevation is almost perfectly linear, since cosine 2A is almost 1.00 at all elevations used. Thus, beyond 30 yards, "gapping" between sighting pins is linear. In other words, if pins are set at 30, 50 & 70 yards, half way between would be 40 and 60 yards. At less than 30 yards, parallax caused by eye-above-arrow causes errors.

When shooting for extreme range, the elevation angle is 45°. Plugging that into Equation #6-12 yields:

$$\cos 2A = \cos 2 \times 45° = \cos 90° = 0.$$

Which is to say that dR/dA = 0 when clout shooting. The fact that the range achieved varies almost not at all for small changes from 45° is well documented. Paul E. Klopsteg made this same observation in 1932 when he said, "Note that 40° gives slightly greater range than 44°, but neither quite reaches the range at 42°." "Since the differences are a matter of only a yard or two, we may say that in this case any elevation between 39° and 44° would give close to maximum range." "This may explain the uncanny accuracy possible in clout shooting."

My own computer run-outs agree generally, but show slightly different observations depending upon the drag/weight ratio of the arrow being shot.

Summary

Although frictionless flight is not what arrows actually experience, it is very useful to compute what a friction-free arrow could do. **No real arrow can do better.** In many circumstances, the difference between frictionless and actual flight is small enough to be ignored.

A few thumbrules can be stated, as follows:

"An arrow can be shot twice as far horizontally as it can be shot vertically."

"Maximum range is proportional to the square of launch velocity."

"Maximum altitude is proportional to the square of launch velocity."

"The time aloft is directly proportional to launch velocity."

"For small angles, range is (almost) directly proportional to launch angle."

Drag Formula:

The amount of drag on an arrow caused by air friction depends upon:

1) The velocity raised to the 1.85th power (approximately).

2) The air density.

3) The areas of the various components of the arrow, specifically:

a) Cross-sectional area.
b) Shaft wall area.
c) Fletching surface area.

In each case, the generic formula is the same, namely,

Drag = coefficient x area x "velocity pressure".

$$D = C \times A \times (VP)$$

Equation 7-1

Velocity Pressure

In every case, the "velocity pressure" is the same, namely:

Velocity pressure equals density times velocity squared divided by 2 times gravity, i.e.:

$$VP = dV^2/2g$$

Equation 7-2

Velocity Pressure Sample Calculation: For an arrow fired at sea level at 200 fps where density is $0.075\#/ft^3$, the velocity pressure is:

$$VP = \frac{0.075\#/ft^3 \ (200ft/sec)^2}{2 \times 32.2 \ ft/sec^2}$$

$$VP = 46.6\#/ft^2.$$

Areas:

The areas to use in Equation 7-1 are, more or less, the surface area of each component; but there are a few minor modifications, as follows:

Tip and nock "areas" are equal to the cross sectional area of the arrow. In the case of a blunt tip, the surface area and the cross sectional area are identical. In the case of a bullet tip, the surface area of the tip is greater than the cross sectional area; yet the cross sectional area is used.

$$A_{tip} = pi \times D^2/4 \quad \&/or$$

$$A_{nock} = pi \times D^2/4$$

Equation #7-3

where: pi = 3.1416
& D = shaft diameter.

The shaft wall area includes the walls of the tip and of the nock.

$$A_{wall} = pi \times D \times L \quad \text{Equation \#7-4}$$

where pi = 3.1416
D = shaft diameter, &
L = length of arrow overall.

The fletching areas include both sides of each fletch.

$$A_{fletch} = LhN \quad \text{Equation \#7-5}$$

where L = length of fletch
h = avg. height of fletch
N = number of sides

Note: N = 6 for a 3-fletch.

Coefficients:

Curiously, the coefficient of friction attributed to each component (tip, shaft, fletching & nock) is calculated in a totally different manner! Postponing the details as to why, allow me to simply state my conclusions as to which formulas apply to each area.

Tip Coefficient

The actual coefficients could reasonably vary between 0.0 and 0.5. My recommendations are as follows:

Field tip .... C_{tip} = 0.15.
Bullet tip ... C_{tip} = 0.05.
Broadheads ... C_{tip} = 0.05*.

* Broadheads are a special case; see discussion later in this chapter.

See Figure 7-2 for the drag of tips at 200 fps for various diameters and various tip coefficients.

Shaft Wall Coefficient:

The correct wall coefficient to use is subject to much argument. I've concluded that the following formula, which I found in Hoerner's book on Pages 2-7 & 2-8, are best. It assumes fully turbulent flow:

$$C_{wall} = \frac{0.074}{Re^{0.2}} + \frac{0.0016L/D}{Re^{0.4}}$$

Equation #7-5

where

L = length of arrow
D = diameter of arrow
Re = Reynold's number

Reynold's Number

The "Reynold's number" is a dimensionless ratio long familiar to mechanical engineers. It is equal to density x velocity x length /viscosity.

$$Re = dVL/ug \qquad \text{Equation \#7-6a}$$

where:

Re = Reynold's number
d = density of air
g = gravity
L = length of arrow
V = velocity of air
u = viscosity of air

When standard conditions for air at sea level are used and when velocity is in ft/sec and length is expressed in inches, the formula for Reynold's number becomes:

$$Re = 515\ V\ L \qquad \text{Equation \#7-6b}$$

Reynold's numbers are used to predict when flow will be turbulent, when it will be laminar, or when it might be either. For flat plates, flow at Reynold's numbers below 100,000 can be expected to be fully laminar regardless of conditions. Between 100,000 and 500,000 flow will be turbulent if triggered; otherwise it will be laminar. Above 500,000 flow will be a mixture of laminar and turbulent unless turbulence has been triggered, in which case flow fill be fully turbulent. See Figure 7-1. Archery fletching Reynold's numbers go from 60,000 to 1,000,000 which means that flow will vary from fully laminar to definitely turbulent. Archery arrow walls' numbers go from 1,400,000 to 8,000,000 which means that flow is always turbulent, either fully or mostly. However, Admiral Moffett's wind tunnel data would indicate fully laminar flow.

I advocate assuming shaft

walls to be fully turbulent and fletching to be laminar and/or (above Re = 500,000) to be at minimum turbulence. But my confidence level is only moderate. I can get good agreement with calibrated flight shots using these assumptions. Yet, Admiral Moffett's wind tunnel tests seem to indicate the very opposite: minimum wall turbulence plus maximum fletching turbulence.

Sample calculation of wall coefficient:

Assume:
Sea level, where
air density is $0.075\#/ft^3$ and viscosity is $3.77 \times 10^{-7}\#sec/ft^2$.
Velocity is 200 ft/sec.
Length is 30" overall.

Using Equation #7-6b:

Re = 515 x 200 fps x 30"

Re = 3,090,000

Using Equation #7-5:

$$C_{wall} = \frac{0.074}{Re^{0.2}} + \frac{0.0016L/D}{Re^{0.4}}$$

$$C_{wall} = 0.074/3{,}090{,}000^{0.2} + 0.0016(30/(21/64))/3{,}090{,}000^{0.4}$$

$$C_{wall} = 0.074/19.9 + (0.0016 \times 91.4)/394$$

$$C_{wall} = 0.003718 + 0.000371$$

$$C_{wall} = 0.004089$$

Although this seems like a small number, it applies to the largest area of an arrow, namely it's shaft walls. Wall drag is greater than all others except when flu-flu fletching is used.

See Table 7-1 for wall drag (in grains) at 200 fps at sea level for any diameter and any length, assuming full turbulence. A maximum deduct is shown if it is felt that flow is not fully turbulent.

Fletching Coefficient

The coefficient for fletching assumes laminar flow. This assumption is wrong for flu-flu fletching but probably correct for all regular fletching:

$$C_{fletch} = 1.328/Re^{0.5}$$

Equation #7-7

where: Re = Reynold's number.

Note: Although the formula for Reynold's number is the same as given in Equation #7-6, the length to use for "L" is the length of the fletching, not the length of the arrow. In fact, the length to use is the average length of the fletch which, for a nominal 5" arrow, is probably about 4.5".

See Figure 7-1 for fletching coefficients at other Reynold's numbers. The lower left line shows fully laminar fletching coefficients. The upper left line show coefficients for fully turbulent flow. Actual flow could be anywhere on or between the two lines. I believe laminar applies.

Sample calculation of fletching coefficient:

Assume:
Sea level, with
Velocity = 200 ft/sec.
Length = 4.5" average.

Using Equation #7-6b:

Re = 515 x 200 fps x 4.5"

Re = 463,500

Using Equation #7-7:

$$C_{fletch} = 1.328/Re^{0.5}$$
$$= 1.328/(463{,}500)^{0.5}$$
$$= 1.328/681$$
$$C_{fletch} = 0.00195$$

Nock Coefficient

The actual nock coefficients could reasonably vary between 0.1 and 1.0. As an educated guess for plastic nocks as commercially common, I recommend using a value of 0.3. Thus:

$$\boxed{C_{nock} = 0.30}$$

See Figure 7-3 for drag of nocks at 200 fps for various diameters and various tip coefficients.

Total Drag

The total drag can now be calculated. The total is the sum of the parts. The item common to all components is the velocity pressure. Thus a single formula for total drag is:

$$D_{total} = (C_{tip}A_{tip} + C_{wall}A_{wall} + C_{fletch}A_{fletch} + C_{nock}A_{nock}) \times VP$$

Equation #7-8

Total drag sample calculation:

Assume previously stated values, which were:

$V = 200$ ft/sec
$L_{shaft} = 30"$
$L_{fletch} = 5"$ nom; 4.5" avg.
$N = 6$ (for 3-fletch)
$h = 1/2"$ fletch height
$D = 21/64"$

Using suggested tip coefficient of 0.15 and using Equation #7-3:

$$C_{tip}A_{tip} = 0.15 \times pi \times D^2/4$$

$$C_{tip}A_{tip} = \frac{0.15 \times 3.14 \times (21in/64)^2}{4}$$

$$C_{tip}A_{tip} = \underline{0.0127\ in^2}$$

Using the wall coefficient previously computed and using Equation #7-4:

$$C_{wall}A_{wall} = 0.00409 \times pi\ D\ L$$

$$= 0.00409 \times 3.14 \times (21"/64) \times 30"$$

$$C_{wall}A_{wall} = \underline{0.1264\ in^2}$$

Using the fletch coefficient previously computed and using Equation #7-5:

$$C_{fletch}A_{fletch} = 0.00195\ LhN$$

$$= 0.00195 \times 4.5" \times 0.5" \times 6$$

$$C_{fletch}A_{fletch} = \underline{0.0263\ in^2}$$

Using the suggested nock coefficient of 0.30 and using Equation #7-3:

$$C_{nock}A_{nock} = 0.30 \times pi \times D^2/4$$

$$= 0.30 \times 3.14 \times (21in/64)^2/4$$

$$C_{nock}A_{nock} = \underline{0.0254\ in^2}$$

Using the above in Equation #7-8:

$$D_{total} = (C_{tip}A_{tip} + C_{wall}A_{wall} + C_{fletch}A_{fletch} + C_{nock}A_{nock}) \times VP$$

From Equation #7-2 & sample calculation:

$$VP = 46.6\#/ft^2$$

$$D_{total} = (0.0127 + 0.1264 + 0.0263 + 0.0254)in^2/144in^2/ft^2 \times 46.6\#/ft^2$$

$$D_{total} = 0.06175\# \times 7000\ grain/\#$$
$$D_{total} = \underline{432\ grains}.$$

Relative Importance of Components

The ratios of drag components are:

	C x A (sq.in)	Drag (gr)	Fraction (%)
Tip	0.0127	24	6.7%
Wall	0.1264	286	66.2%
Fletching	0.0263	60	13.8%
Nock	0.0254	57	13.3%
Totals	0.1908	432gr	100.0%

See Figure 7-2 for tip and/or nock drag (in grains) at 200 fps at sea level for any diameter and any coefficient.

See Table 7-1 for wall drag (in grains) at 200 fps at sea level for any diameter and any length, assuming full turbulence. A maximum deduct is shown if it is felt that flow is not fully turbulent.

See Table 7-2 for fletching resistances at sea level at 200 fps.

Table 7-2 ... Fletching Resistance at 200 ft/sec.

Length (nominal)	Number	Laminar Drag	Turbulent Drag
3"	3-fletch	**40**	94
4"	4-fletch	**63**	163
5"	3-fletch	**54**	151
Broadheads:			
1"	2-blade	**19**	34
1"	3-blade	**28**	50

Assumptions:

Fletch length is 1/2" shorter than nominal.
Area of one side is 0.45" x (L-½").
Broadhead length = 1.0".
Broadhead surface area (both sides) is 1.0 sq.inch per blade.
Suggested drag is "laminar".

Drag at other velocities:

For small changes of velocity, it is reasonably accurate to assume that the coefficients of drag remain unchanged and that therefore the drag is proportional to the square of the velocity. In actual fact, the coefficients for tips and nocks do remain constant. The coefficients for walls and fletching depend, however, upon the Reynold's numbers which, in turn, depend upon velocity. The wall coefficient is, per Equation #7-5, mainly dependent upon Reynolds number raised to the -0.2th power. Thus the resistance of the wall depends upon velocity raised to the 2.0th - 0.2th = 1.8th power. A similar analysis for fletching yields 2.0th - 0.5th = 1.5th power. Weighting each according to the total for our sample arrow yields:

Tip	6.7% x 2.0th =	13
Wall	66.2% x 1.8th =	119
Fletching	13.8% x 1.5th =	21
Nock	13.3% x 2.0th =	27
Totals:	100%	180

Average "power" = 1.80th.

Thus this arrow's drag is proportional to it's velocity raised to the 1.80th power. For a general rule, 1.85 is the power I use, based upon similar calculations for many arrows. Thus the following formula for converting drag at 200 ft/sec to drag at any other speed:

$$\text{Drag} = (\text{Drag @ 200fps}) \times (\text{Velocity}/200\text{fps})^{1.85}$$

Equation #7-9

Table 7-3 ... Speed Multipliers

Velocity (ft/sec)	Drag Force Multiplier
300 fps	2.117
280 fps	1.864
260 fps	1.625
240 fps	1.401
220 fps	1.193
200 fps	**1.000**
180 fps	.823
160 fps	.662
140 fps	.517
120 fps	.389
100 fps	.277

Terminal Velocity

An arrow's free-fall terminal velocity says more about it's flight characteristics than does drag alone. The total drag on an arrow equals the arrow's weight when the arrow is traveling at it's terminal velocity. Thus the author recommends that the arrow's terminal velocity be computed immediately following the calculation of the arrow's total drag. The total drag applies to only one speed. The terminal velocity remains unchanged regardless of speed. The formula for calculating terminal velocity is obtained by re-working the previous formula and substituting the arrow's weight for the unknown drag. The result is:

$$V_{,terminal} = V_{drag}(W_{,arrow}/Drag)^{(1/1.85)}$$

Equation 7-9

where

$V_{,terminal}$ = terminal velocity
$V_{,drag}$ = velocity at which drag was computed.
$W_{,arrow}$ = weight of arrow
Drag = drag computed at particular velocity.

Sample Calculation of Terminal Velocity:

The total drag calculated for a 2117 x 30" arrow was 432 grains. The weight of a 2117 x 30" arrow is 551 grains, according to Easton. Using Equation 7-9:

$$V_{,terminal} = 200fps(551\ gr / 432\ gr)^{(1/1.85)}$$

$$V_{,terminal} = 200fps \times 1.28^{0.54}$$

$$V_{,terminal} = 200\ fps \times 1.14$$

$$V_{,terminal} = 228\ fps.$$

This terminal velocity of 228 fps for a 2117 x 30" arrow has been confirmed by author's experiments, as described in chapter on Calibrated Flight Shooting.

Misaligned Fletching

The calculation of drag across regular fletching assumes that the fletching is rigid and that all fletches are aligned in such a way that the air passes over them without having to change direction. Flu-flu fletching does not satisfy this assumption because it is not rigid. Flu-flu will be discussed later. For now, let's address the question of misaligned fletching.

An example of misalignment would be a case where two of three fletches were aligned straight and the other was aligned helically. The air passing by obviously has to change direction. The drag caused by this is not a function of skin friction, much as tip and nock drag is unrelated to skin friction. Rather, it is related to how much turbulence is left behind. There is no way to accurately predict how much drag will be caused, but an approximation can be made. The basic formula given in Equation #7-1 applies. Namely, drag =

coefficient x area x velocity pressure. The coefficient is not apt to exceed 1.0; nor is it apt to be much less than 1.0, although it could be. The "area" is the hard thing to define. The area is the cross-section area of disturbed air. In the case of the example where two of three fletches were straight and the third was helical, the area would be that seen when looking along the axis of the arrow. A calculation on the assumptions that the coefficient equals 1.0 and the areas equal, for instance, 1/2" high x 1/8" "wide" yields:

$$D = C\ A\ VP$$

$$= 1.0 \times 1/16\text{th in}^2 \times VP$$

$$= 0.0625\ \text{in}^2 \times VP$$

Comparing that number to those computed for a 2117 x 30 regular arrow shows that the drag caused by misalignment could be quite large.

Broadhead Rotation

Fletching is spiral on broadhead-tipped arrows. Fletching for broadheads must be spiraled in order to "average out" the steering and/or planing effect of the broadhead. The broadhead itself is not spiraled. Thus the fletching and the broadhead are fighting one another. The broadhead wants no rotation. The fletching wants the rotational speed that it has when flying with a target tip. There is a certain amount of energy dissipated by the fight between fletching and broadhead.

I've seen archers with freely rotating broadhead blades. Doubtlessly, a freely rotating broadhead will not only lose less energy in flight but will steer better as well.

To calculate the amount of energy involved would be difficult. The formula would be: energy loss = torque x rotational speed. The rotational speed can be identified rather easily, but the torque would be much tougher to predict. None the less, every archer who spins his arrows (to check for straightness) intuitively knows that the amount of torque caused by a broadhead paddling the air is very small.

Flu-flu drag

Flu-flu fletching is not only misaligned, but it also flaps like a flag. The author tested the flu-flu fletching shown in the photo. The test was as described in this book's chapter on "Calibrated flight Shooting".

Resistance of the flu-flu fletching as 2.8 times greater than would be predicted for flat plates. According to Sighard F. Hoerner in "Fluid-Dynamic Drag" on page 3-25, flags have greater resistance to air flow than do flat plates by factors of 10. Since only a portion of the flu-flu fletching is flapping like a flag and the rest of it is stiff like regular fletching, the 2.8 increase seems reasonable. At the same time, it shows that an accurate prediction of flu-flu resistance is difficult.

High Speed Flutter

Last night I was watching Tink Nathan in his Adventure Video on bowhunting in Australia. He happened to mention that some archers are experiencing fletching flutter as a consequence of the higher velocities being achieved these days. That comment tied in directly with the above discussion of the fact that a flapping flag has about 10 times as much resistance as a rigid plane. It makes sense. Personally, I haven't been fortunatel enough to achieve those 250 fps plus speeds.

Incidently, Tink Nathan sells

"Star Flight" fletching which tapers symetrically front and rear. It makes sense that such a shape would be structurally more resistant to flutter.

Parachute Drag

Were one to add a parachute to the end of his arrow, the drag formula would be same as for a tip or nock, but the coefficient would always be about 1.0. The fluffy little adders some hunters put on their arrows to keep the arrow from passing all the way through a turkey, are like parachutes. Assuming a coefficient of 1.0, the formula for any type of parachute drag becomes:

Drag = area x velocity pressure.
Equation 7-10

where area is as "seen" by the airstream.

Tip Force: Amplifying Comments.

Earlier in this chapter I said that the coefficients to use for tip force were between 0 and 0.5, but I did not say why. The amount of turbulence created by the parting of air at the tip of an arrow is what determines the amount of energy lost. Any reasonably steamlined tip will create almost no turbulence as it gently parts the air. A blunt tip might be expected to create almost total turbulence, which would result in a coefficient of 1.00. As it happens, though, even a blunt tip is not that bad. What happens is that a bubble of air builds up ahead of the blunt tip, making the combination of blunt plus bubble act much like a dome-tipped arrow. The lack of importance of the tip's shape is evident in aircraft design. Look an most any commercial jet and you will see a dome-shaped nose. I would guess that the coefficients for either a dome or a bullet tip would be very close to zero. For a field tip, which has discontinuities in it's shape, I assume a 0.30 coefficient.

Nock Drag

The tail end of an arrow doesn't actually drag. Rather, it fails to restore full atmospheric pressure. A totally blunt tail end would leave almost total turbulence in it's wake. It's coefficient would be 1.00. A very streamlined, pointed tail would leave very little turbulence. It's coefficient would be near zero. As it happens, it is much easier to part air without turbulence than it is to put it back together again without turbulence. A dome-shaped tail is almost totally useless in re-assemblying the air. Look at any commercial jet aircraft and you will see a long, slender tail. That is what is needed to re-join the air stream with minimum turbulence. I simply guess that the nock of an arrow has a coefficient of 0.30. It may be worse that that.

Tip + Nock

The literature always considers the front and rear end of an object simultaneously. It was my own idea to consider them separately. In the case of an arrow, what happens at the tip is so far removed from the tail that it struck me that they should be evaluated independently. Were tip and nock considered simultaneously, the worst case would be where the tip had a coefficient of 1.0 and the tail had the same, for a total of 2.0. In Vennard at a Reynold's number of 34,000 I found the following combined coefficients:

1.90 ... flat plate of infinite length traveling sideways.

1.20 ... for a cylinder going sideways

1.20 ... for a flat disc.

0.50 ... for a sphere.

0.08 ... for a streamlined strut.

Of the shapes mentioned, the tip plus the nock of an arrow probably most closely resemble a sphere, which has a coefficient of 0.50. My assumptions of 0.15 for a field tip plus 0.30 for a nock would be 0.45, which is close to that of a sphere's 0.50. I throw tnis in just to lend a little weight to my guesses.

Steering Drag

All the preceding analysis in this chapter assume that the arrow is flying straight. It does not tell what the drag consequences of a porpoising or fishtailing arrow might be. Even a straight flying arrow has to be steered. An arrow shot on the moon will hit with the same up-angle it had when launched. Fletching on the moon would be useless. An arrow in air has to be continuously re-aligned as it arches through the air.

William Bollay's "Air Resistance of Trains, Automobiles and Ships" copies Schmidt, (ZVdI, 82, 1938, p. 188) to show that the resistance of a non-streamlined train may be doubled when the relative wind comes from an angle of yaw, although that of the streamlined train remains practically constant. I throw this out simply as an area for further study.

I would suggest the following experiment and may, if I get the time, actually conduct it:

Put an arrow velocity meter down range 20 or 30 yards. Shoot arrows from a well-tuned bow. Note the velocity. Shoot same arrows from same bow but locate nocking point so as to cause fish-tailing. Note loss of velocity, if any. Any loss is due to fishtailing. Repeat with arrow meter close to launch point to confirm that velocity loss happens during flight, not during launch.

Boundary Layers

When air flows along a surface, a boundary layer builds up between the surface and the remote, undisturbed air. I wasn't able to find any prediction of what the thickness of a boundary layer around an arrow shaft would be, but the thickness of boundary layers over flat plates were well documented. For laminar flow the formula is:

$T = 5.2L/Re^{\frac{1}{2}}$

& for turbulent flow it is:

$T = 0.37L/Re^{0.2}$

where:

T = thickness, with same dimensions as "L".
L = length from starting point.
Re = Reynold's number.

For an arrow's turbulent flow along it's wall, the thickness computes to be greater than 1/2". Thus the fletching would be totally engulfed in boundary layer. What that might do to the calculation of fletching resistance is unknown to me. I would guess, though, that it would reduce the fletching resistance because the air passing over the fletching is being dragged along with the arrow, reducing it's relative velocity over the fletching.

Wind Tunnel Data

Rheingans reported that Rear Admiral Moffett (of Moffett Naval Air Station fame) conducted a wind tunnel experiment on an arrow the following characteristics:

Head = ogival.
Length = 26".
Diameter = 5/16".
Fletch = 3-feather x 2½" with
total area = 7.5 square inches
Velocity = 200 ft/sec.

With feathers removed:

Drag = 112 grains.

With feathers on:

Drag = 273 grains.

Comparison with predicted for the shaft without feathers is:

Tip (.05), Fig.7-2 =	9 gr.
Nock (.3), Fig.7-2 =	52 gr.
Shaft, Tbl 7-1 =	242 gr.
deduct, Tbl 7-1 =	-37 gr.
Sub-total =	266 gr.
Fletch, Tbl 7-2	44 gr.
Total =	310 gr.

This result is rather amazing in that the total resistance is not too far off, but the components are way off.

Back-calculating the bare shaft result forces one to the conclusion that flow along the shaft was laminar. Yet the literature says that flow should definitely be turbulent. Perhaps in the quiet wind tunnel environment there was no "trigger" to start the turbulence.

Back-calculating the fletching resistance, which was 272-112 = 160 grains and comparing it to the 44 grains predicted leads to the conclusions that flow was not only fully turbulent but has resistance 50% higher than full turbulence! I strongly suspect that misalignment caused this high resistance. An arrow in free flight will tend to rotate at whatever speed gives best alignment. Chances are the admiral had to hold the arrow rigidly in place, preventing it from rotating and thus aligning.

Summary Comments

In studying the theory of aerodynamics and of fluid drag I become aware of how very complicated the topic is. Repeatedly the authors tell the reader that tests have to be conducted to verify. Drag coefficients do surprising things, such as suddenly dropping or suddenly increasing. The Reynolds numbers applicable to archery are right in the middle of where transitions between laminar and turbulent occur. The strange things that happen include:

In aircraft, the use of fillets at the junctions of wing and body make a huge difference. The same concepts might apply to fletching.

Little bumps to cause boundary layer tripping are used in aircraft; maybe the concept would apply to arrows.

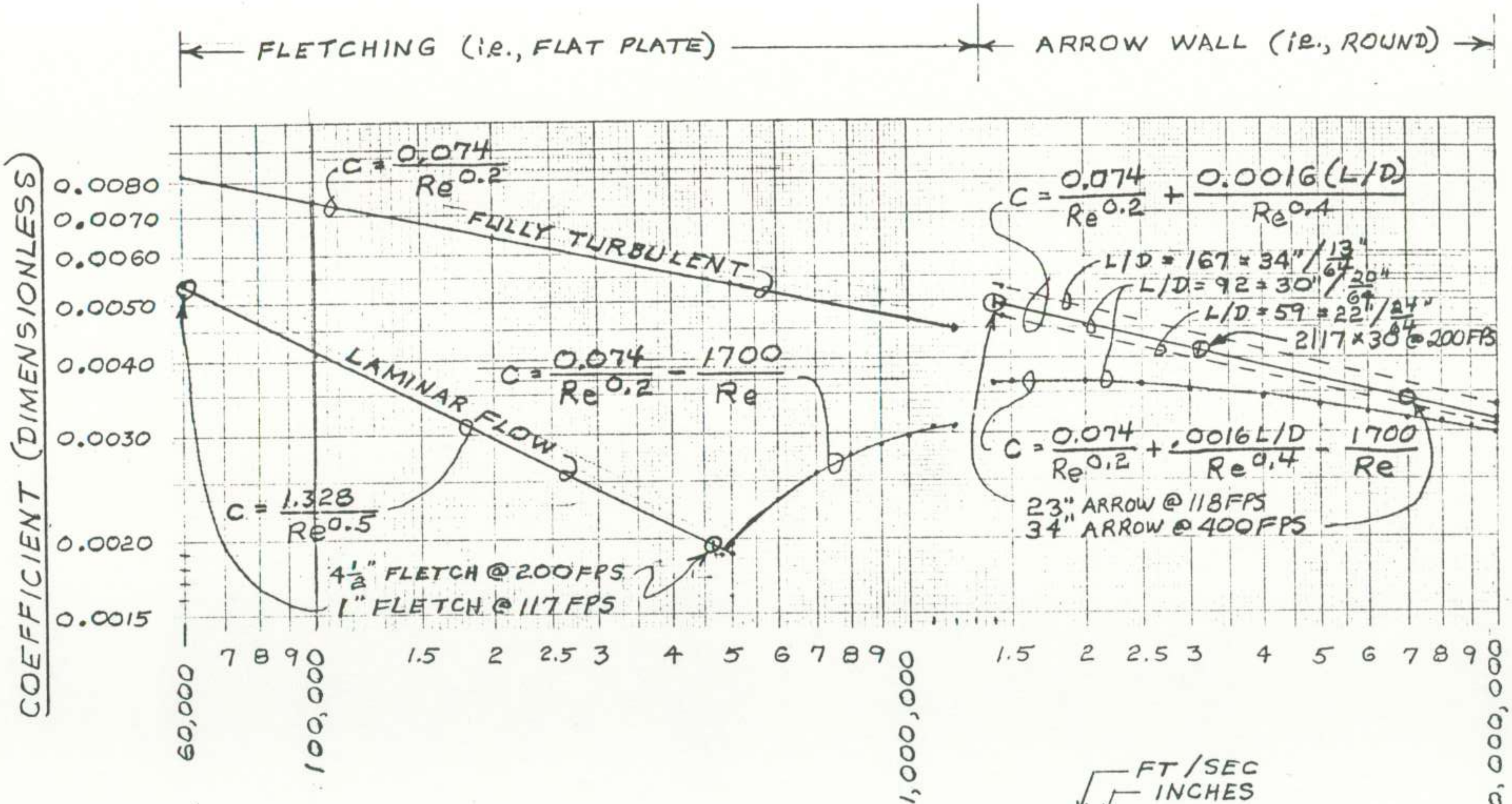
Removal of boundary layer so as to keep flow laminar is done in aircraft. Perhaps a perforated shaft with open rear end could do the same.

Dimpled golf balls have only a tiny fraction of the resistance that smooth spheres have; perhaps dimpled arrows would do the same.

The boundary layer thickness of the arrow shaft upon arrival at the fletching may or may not engulf part or all of the fletching.

There is a lot of room for further study!

*** END ***



REYNOLD'S NUMBER $= Re = \frac{V X \rho}{\mu g} = 515\,VL$ (SEA-LEVEL AIR)

Figure 7-1 ··· DRAG COEFFICIENTS FOR FLETCHING & SHAFTS

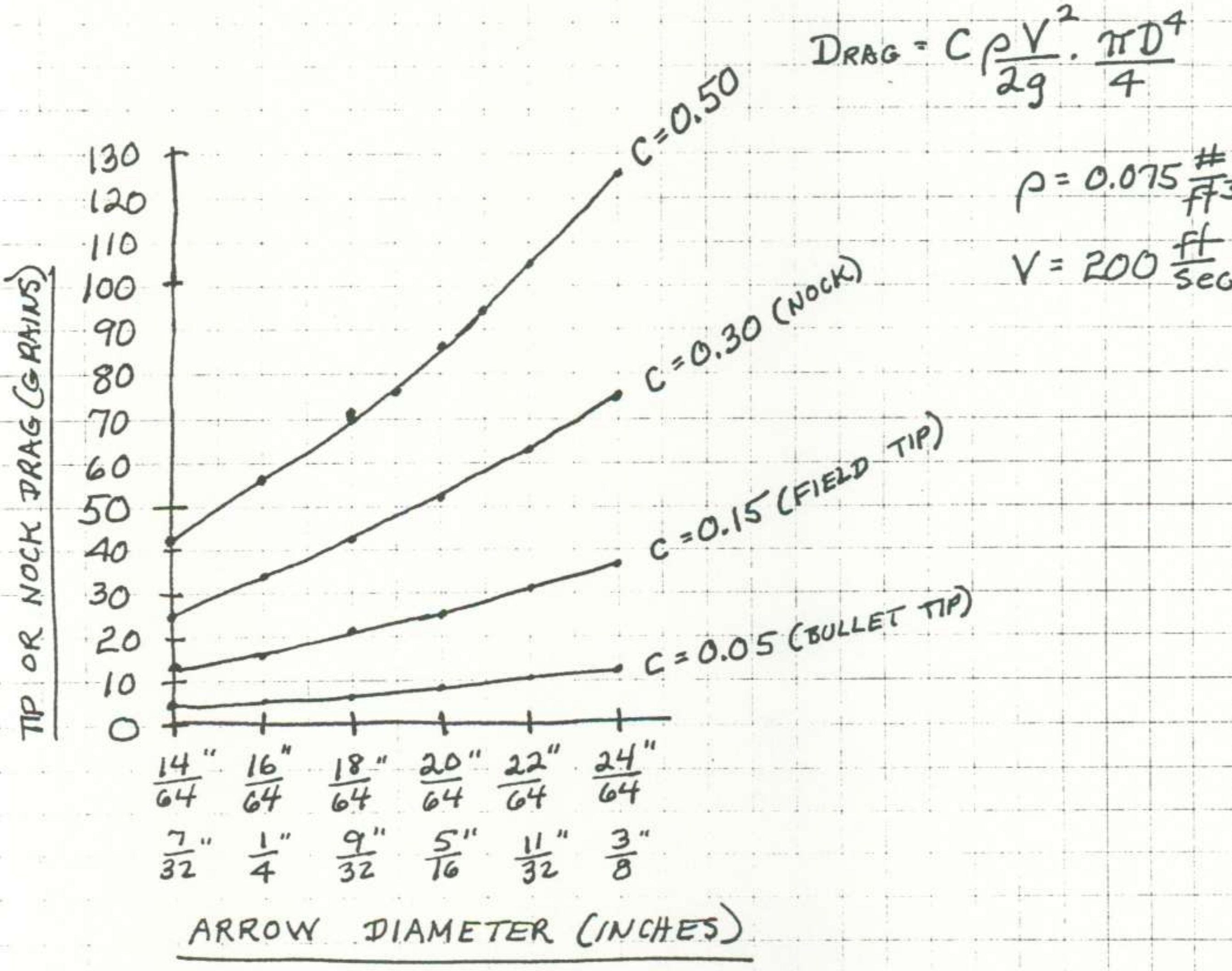


Figure 7-2 … TIP or NOCK DRAG @ 200 FPS

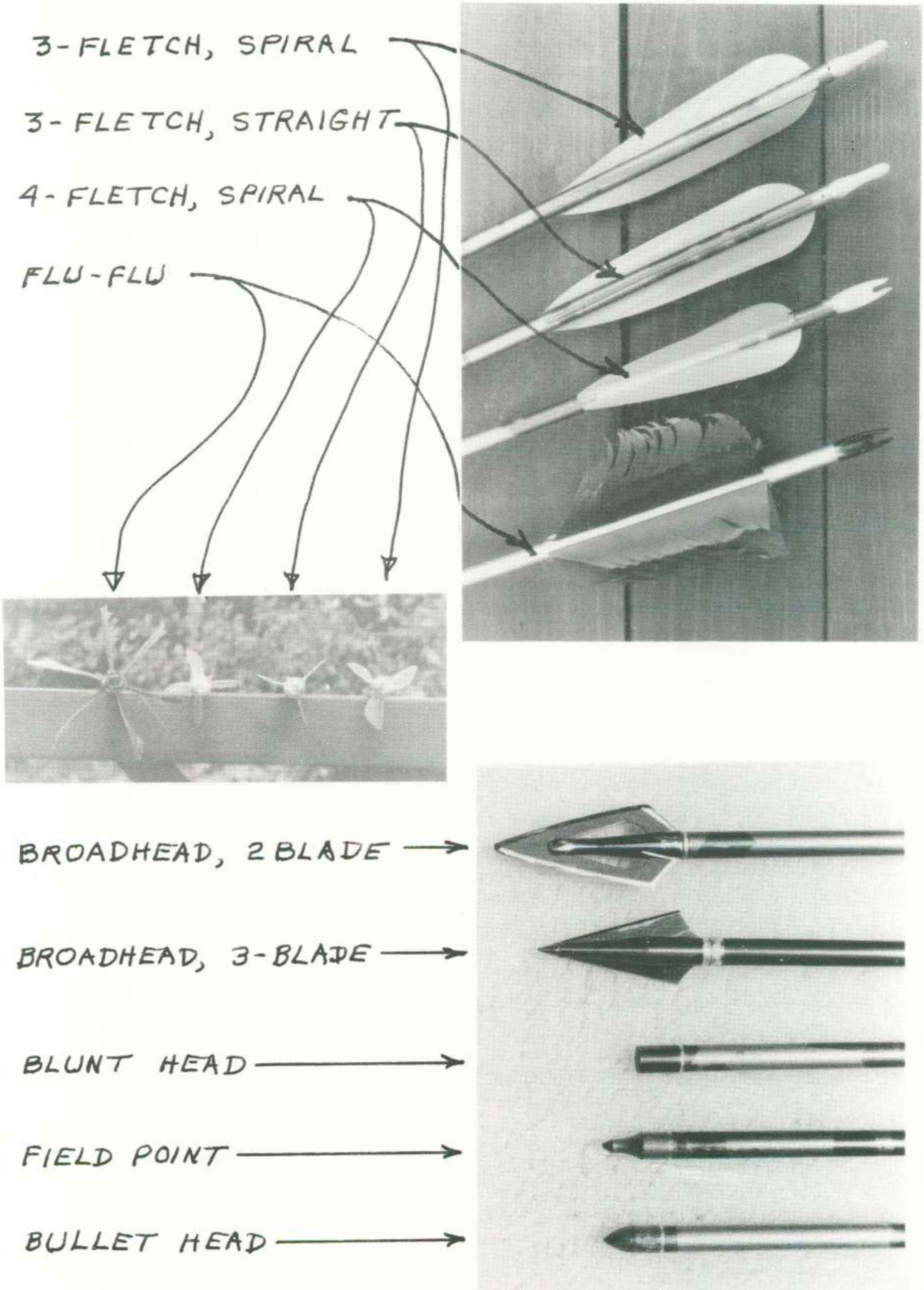


Figure 7-3 ... FLETCHING & TIPS

Table 7-1 ... WALL DRAG OF ARROWS (grains) @ 200 FPS

ARROW'S LENGTH (inches)	ARROW'S DIAMETER (inches) 13"/64	7"/32	15"/64	1"/4	17"/64	9"/32	19"/64	5"/16	21"/64	11"/32	23"/64	3"/8
22"	142	152	161	171	181	190	200	210	219	229	239	249
	118	126	134	142	149	157	165	173	181	189	197	204
	28	30	32	35	37	39	41	43	45	47	50	52
23"	148	158	168	178	188	198	208	218	228	238	248	258
	124	132	140	148	157	165	173	181	189	198	206	214
	29	31	33	35	38	40	42	44	46	49	51	53
24"	153	164	174	184	195	205	216	226	236	247	257	268
	129	138	147	157	167	178	188	199	209	219	230	240
	29	32	34	36	38	41	43	45	47	50	52	54
25"	159	170	180	191	202	213	223	234	245	256	266	277
	135	144	153	162	171	180	189	197	206	215	224	233
	30	32	35	37	39	41	44	46	48	51	53	55
26"	165	176	187	198	209	220	231	242	253	264	276	287
	141	150	159	169	178	187	196	206	215	224	233	243
	30	33	35	38	40	42	45	47	49	52	54	56
27"	170	182	193	205	216	228	239	250	262	273	285	296
	147	156	166	175	185	195	204	214	223	233	242	252
	31	33	36	38	41	43	45	48	50	53	55	57
28"	176	188	200	211	223	235	247	258	270	282	294	305
	152	162	172	182	192	202	212	222	232	242	252	261
	32	34	37	39	41	44	46	49	51	54	56	58
29"	182	194	206	218	230	242	254	266	279	291	303	315
	158	168	178	189	199	209	220	230	240	250	261	271
	32	35	37	40	42	45	47	50	52	55	57	59
30"	188	200	212	225	237	250	262	274	287	299	312	324
	164	174	185	195	206	217	227	238	248	259	270	280
	33	35	38	40	43	45	48	50	53	55	58	60
31"	193	206	219	231	244	257	270	283	295	308	321	334
	169	180	191	202	213	224	235	246	257	268	279	289
	33	33	33	33	33	33	33	33	33	33	33	33
32"	199	212	225	238	251	264	277	290	304	317	330	343
	175	186	198	209	220	231	243	254	265	276	288	299
	34	36	39	42	44	47	49	52	55	57	60	62
33"	205	218	231	245	258	272	285	298	312	325	339	352
	181	192	204	215	227	239	250	262	273	285	296	308
	34	37	40	42	45	48	50	53	55	58	61	63
34"	210	224	238	251	265	279	293	306	320	334	348	361
	186	198	210	222	234	246	258	270	282	294	305	317
	35	38	40	43	46	48	51	54	56	59	62	64

Highest drag number assumes full turbulence flow.
Middle drag number assumes partial turbulence flow.
Smallest drag number assumes fully laminar flow.

Three different sets of calculations apply for three different types of trajectories:

a) Friction-free trajectories, see Chapter 6.

b) Trajectories with friction & small launch angles.

c) Trajectories with friction & large launch angles.

The friction-free trajectories have formulas to totally describe them, as given in Chapter 6, "Trajectories, Frictionless". Trajectories with friction and small launch angles (up to about 6°) can be calculated by ignoring vertical friction and using an exact formula for horizontal friction. Trajectories with friction and large launch angles (more than 6°) have to be broken into multiple small segments. This is practical using computers but is not practical using long hand.

TRAJECTORIES WITH FRICTION & SMALL LAUNCH ANGLES

I tried but failed to derive equivalent exact formulas for flight with friction. I was, however, able to come up with the exact description of horizontal deceleration. Using the exact formula for horizontal deceleration with-friction plus the exact formulas given in Chapter 6 for friction-free vertical accelerations gives nearly exact results when used with small launch angles. In all ordinary archery, the amount of vertical speed loss caused by air drag is very small. For example, when shooting 100 yards with a 200 fps arrow, the initial vertical speed is about 24 fps. Halfway to the target vertical speed is zero. Average vertical speed is 12 ft/sec. The relative importance of friction in the horizontal and vertical directions is :

$$\left(\frac{198.5\,fps}{12.0\,fps}\right)^{1.85} = 16.54^{1.85} = 180$$

Thus vertical friction can be ignored with negligible error when shooting horizontally at ranges under 100 yards.

For the engineer or mathematician who wants to take up where I left off, the differential formula for vertical travel with friction, which I am unable to integrate is:

$$dV_y = \frac{-g}{V_y}\left[\frac{k}{w}\cdot V_y^{1.85} - 1\right] dy$$

The introduction of gravity and the deceleration to zero upward velocity, and the subsequent acceleration downward to terminal velocity is quite beyond my ability to integrate.

Horizontal Deceleration

The slowing of a ship which has lost power or of a car that has been put in neutral or of an arrow which has magically been prevented from dropping can be computed from the formula:

$$V = \left(V_o^{0.15} - \frac{0.15\,k\,x}{m}\right)^{\frac{1}{0.15}}$$

Equation #8-1

where V = velocity at point "x"

V_o = velocity, original

k = constant,

drag = $Df/Vo^{1.85}$

m = mass

This formula was developed for me by Assistant Professor Van P. Carey of the University of California Mechanical Engineering Department in June 1986. I had developed the following differential formula but was unable to integrate it:

$$dV = (-k/m)V^{0.85}dx$$

That formula was, in turn, derived from the following as follows:

(1) $dx = Vdt$, a definition of velocity

(2) $dV/dt = D_F/m$, from law of physics: $F = ma$

(3) $D_F = k V_o^{1.85}$, from drag force analysis in previous chapter.

Eliminating D_F from the last two yields:

$dV/dt = (k/m)(V^{1.85})$, which says that deceleration is proportional to velocity to the 1.85 power.

Eliminating dt from the first and last formulas yielded:

$dV = \left(\frac{-k}{m}\right)V^{0.85}dx$ which says that velocity loss per yard is proportional to velocity to the 0.85 power.

The professor integrated this to get

$$V = \left(V_o^{0.15} - \frac{0.15\,Kx}{m}\right)^{\frac{1}{0.15}}$$

Equation #8-1.

Energy Loss:

The problem with using Eqn (8-1) while meaning to ignore vertical friction is that energy loss is proportional to the square power of total velocity, not just the horizontal component of total velocity. Midway through the arrow's flight, however, the total velocity and the horizontal component are the same. In the interest of simplicity, the error introduced shall be ignored. At worst, an error of about 1% less energy loss than actual will prevail for the sample case of a 200 fps arrow at 100 yards, 358 grain drag, 546 grain weight.

Sample calculation of speed & energy loss:

Arrow is per sample in Chapter 7, which was: 2117, 28" long, with field point.

$V_o = 200$ ft/sec.

Drag = 386 grains of force

Weight = 527 grains of weight.

$$k = D/V_o^{1.85} = \frac{386 \text{ grains}}{(200 \text{ ft/sec})^{1.85}}$$

$$V = \left[\left(200\frac{ft}{sec}\right)^{0.15} - \frac{0.15 \times 386gr \times 32.2\frac{ft}{sec^2}}{\left(200\frac{ft}{sec}\right)^{1.85} \times 527gr}\right]^{\frac{1}{0.15}}$$

$$V = (2.214 - 0.000196\,x)^{\frac{1}{0.15}}$$

For x = 10 yds = 30 ft,

$$V_{10\ yd} = (2.214 - 0.000196 \times 30)^{\frac{1}{0.15}} = 2.21^{\frac{1}{0.15}}$$

$V_{10\ yd}$ = 196.5 ft/sec

Velocity loss = $V_o - V_{10\ yd}$

= 200.0 - 196.5 = 3.5 ft/sec

Energy loss, percent =

$$100\% \left(V_o^2 - V_{10yd}^2\right)/V_o^2 = 3.5\%$$

Repeating at ten yard intervals and tabulating yields the following table.

TABLE 8-1

TABLE OF ENERGY & VELOCITY LOSS VS. RANGE

Arrow = 2117, 28" long, 527 grains weight, 386 grains drag at 200 ft/sec, 5" plastic 3-fletch.

Range	(yards)	0	20	40	60	80	100
Velocity	(ft/sec)	200	192.5	186	180	173	167
Specific energy	(ft-lb/lb)	621	579	539	501	466	434
Velocity loss	(%)	0	3.5	6.9	10	13.3	16.4
Energy loss	(%)	0	6.8	13.3	19.3	24.9	30.2

ELEVATION ANGLES WITH FRICTION

Back in Chapter 6 the elevation angle needed to fire a frictionless arrow at a horizontal target was given as:

$A = \frac{1}{2} \arcsin (Rg/V^2)$,
Equation #6-11

A similarly clean formula would be nice to have for elevation angles with friction, but this engineer hasn't found one. However, by combining the formula worked out earlier for horizontal friction, which was:

$$V_x = \left(V_{0,x}^{0.15} - 0.15\,kx/m\right)^{\frac{1}{0.15}}$$

Equation #8-1

with the formula for frictionless vertical travel, which was:

$y = t\,V_o \sin A - \frac{1}{2} g t^2$,
Equation #6-1

One can compute the necessary elevation angle with friction. The approach to use is as follows: First find out how long it will take for the arrow to travel the distance to the target. Then figure out what elevation angle is needed to keep the arrow aloft that long. The answer is computed from:

Eqn (6-5): $t = (2\,V_o \sin A)/g$. Solving for A yields:

Eqn (8-2): $A = \arcsin (tg/2V_o)$

Procedure for Calculating Time of Flight & Elevation Angle

1st: Compute elevation angle for frictionless flight from:
Eqn (6-11): $A = \frac{1}{2} \arcsin (Rg/V^2)$

2nd: Compute initial horizontal component of velocity from:
$V_{o,x} = V_o \cos A$

3rd: Compute 1st approximation of time of flight, ignoring friction, from:

$t = R/V_{o,x}$

4th: Compute velocity upon arrival at target using:

$$V_R = \left[(V_o \cos A)^{0.15} - \frac{0.15\,kx}{m}\right]^{\frac{1}{0.15}}$$

5th: Compute average horizontal velocity using $V_x = \frac{1}{2}(V_{o,x} + V_R)$

6th: Compute 2nd approximation of time aloft from:
$t = R/V$

7th: Compute elevation angle from: Eqn (8-2): $A = \arcsin tg/2V_o$

8th: Repeat steps 2 thru 7 if needed for greater accuracy using angle computed in step 7 above.

Sample Calculation

Arrow = 2117, 28" long, 200 ft/sec, weighing 527 grains and having 386 grains of drag at 200 fps. Compute 100 yard elevation angle.

1st: $A = \frac{1}{2} \arcsin Rg/V^2$

$$= \frac{1}{2} \arcsin \left[\frac{300 \text{ ft} \times 32.2 \text{ ft/sec}^2}{(200 \text{ ft/sec})^2}\right]$$

$$= \frac{1}{2} \arcsin 0.2415 = \frac{1}{2} \times 13.98° = 6.99°$$

2nd: $V_{o,x} = V_o \cos A = 200 \frac{\text{ft}}{\text{sec}} \cos 6.99°$

$$= 198.51 \text{ ft/sec}$$

3rd: $t = R/V_{o,x}$

$$= 300 \text{ ft}/198.51 \text{ ft/sec}$$

$$= 1.511 \text{ sec}$$

4th: $$V_R = \left[\left(198.51 \frac{\text{ft}}{\text{sec}}\right)^{0.15} - \frac{0.15 \times 386 \text{ gr} \times 32.2 \frac{\text{ft}}{\text{sec}^2} \times 300 \text{ ft}}{527 \text{ gr} \times \left(200 \frac{\text{ft}}{\text{sec}}\right)^{1.85}}\right]^{\frac{1}{0.15}}$$

5th: $$V = (2.2114 - 0.05874)^{\frac{1}{0.15}}$$

$$= 165.9 \text{ ft/sec}$$

6th: $t = R/V$

$$= \frac{300 \text{ ft}}{182.2 \text{ ft/sec}} = 1.647 \text{ sec}$$

7th: $A = \arcsin tg/2V_o$

$$= \arcsin \frac{1.647 \text{ sec} \times 32.2 \text{ ft/sec}^2}{2 \times 200 \text{ ft/sec}}$$

$$= \arcsin 0.265 = 7.62° = A$$

(first try)

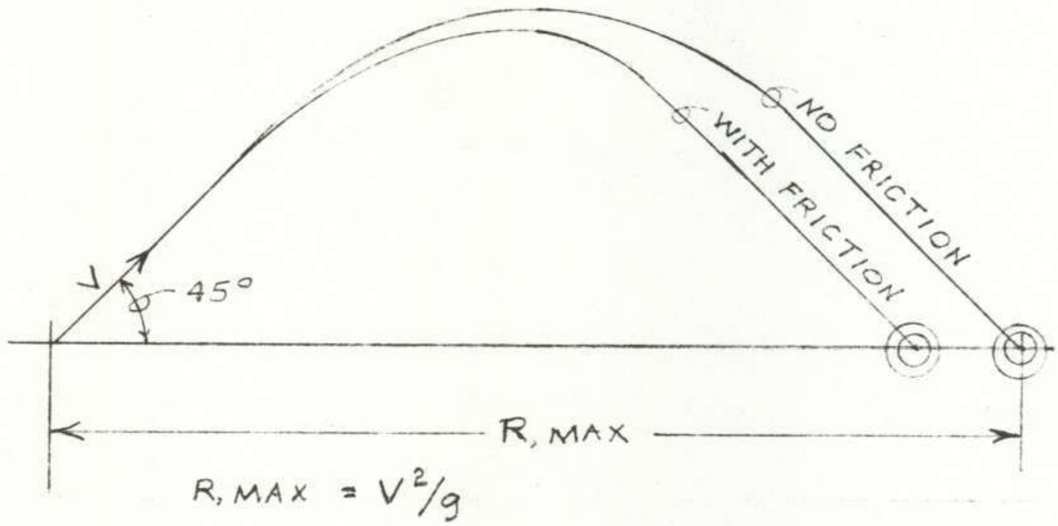
8th: Repeat and compute $A = 7.63°$ (final try)

Pin Sights

One reason I wanted to know the elevation angle for any given range was so as to know what the relative spread between sighting pins should be. The distance between pins (for ranges beyond 30 yards and up to 100 yards) should be proportional to the differences in elevation angles. In Chapter 6, for frictionless flight, the formula was given as:

Eqn (6-12): $dR/dA = (-2V^2/g)\cos 2A$

There is no equally convenient formula for flight with friction, so data must be tabulated. See Table 8-2. Studying the "difference" data for elevation angles with and without friction shows that "gapping" between sighting pins is hurt by friction. For instance, the angular difference between 20 yd and 40 yd without friction is 1.388°; between 40 yd and 60 yd it is 1.394°, a ratio of 1.004, or 4/10%. The numbers with friction are 1.462°, 1.519°, 1.039, or 3.9%. Even so, interpolation between pins is accurate enough for all practical purposes beyond 30 yards.



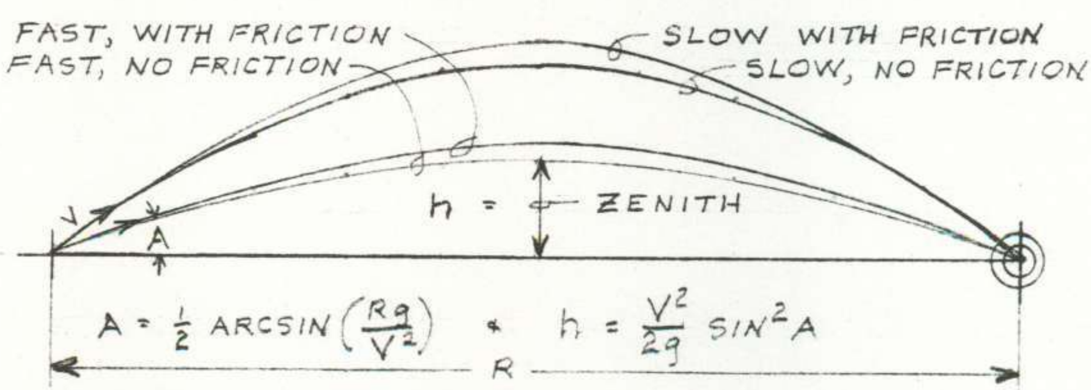


Figure 8-1 ... Trajectories

TABLE 8-2 ... ELEVATION ANGLES WITHOUT & WITH FRICTION

Arrow is 2117 with 5" 3-fletch with field point weighing 527 grains and having drag of 386 grains at an initial velocity of 200 ft/sec.

Range (yds)	0	20	40	60	80	100
Without Friction:						
Elevation angle (deg)	0	1.384°	2.772	4.166	5.570	6.988
Time of flight (sec)	0	0.300	0.601	0.902	1.206	1.511
Difference		1.384°	1.388°	1.394	1.404°	1.418°
With Friction:						
Velocity, arrival (fps)	200	192.97	186.03	179.17	172.38	165.66
Time of Flight (sec)	0	0.305	0.622	0.951	1.293	1.649
Elevation angle (deg)	0°	1.409°	2.871	4.390	5.973°	7.628°
Difference		1.409°	1.462	1.519°	1.583°	1.655°

Under 30 yards, parallax caused by height of eye above arrow causes errors.

As an example, half way between the 40 yard pin and the 60 yard pin should be the 50 yard pin. Running the numbers out shows that half way between is actually 50.10 yard, which is close enough.

TRAJECTORIES WITH FRICTION & LARGE LAUNCH ANGLES

If there exists a simple formula for calculating directly the motion of a projectile with friction, I could neither find it nor derive it. The only way to get the answers, then, is to divide the flight into little parts and compute the entering and leaving conditions of each part. For satisfactory accuracy I found it necessary to divide the arrow's flight into intervals of 1/20th of a second. Even with a computer, there are an awful lot of computations involved. The program I wrote for a Hewlett-Packard HP41CV hand held programmable calculator with printer (published herein) takes about 30 minutes to compute a single flight. Worse, it takes trial and error to get the data for a particular flight. Say, for instance, that you want to know all about a flight of 220 yards. You have to guess at a launch angle and then compute until the arrow hits the ground at some range. Then try another angle and do it over again. I've done this and published the results in Figure 8-2. The mathematical procedure is as follows, with sample calculation:

Enter Starting Data:

V,o = velocity @ launch = 200 ft/sec

A,o = angle of launch = 45°

W = weight of arrow = 527 grains

D,o = drag at launch velocity = 386 grains

TD = time increment for each segment

= 0.05 seconds

Calculate Drag

D = drag at beginning of interval

$D = D,o(V/V,o)^{1.85}$

$= 386\ gr\ (200/200)^{1.85}$

= 386 grains

Compute Accelerations:

a = acceleration due to drag

$a = (D/W)g = \left(\frac{386\,gr}{527\,gr}\right) 32.2 \frac{ft}{sec^2}$

$= 23.58 ft/sec^2$

a,x = acceleration due to drag, horizontal component

a,x = a cos A

$= 23.58 \frac{ft}{sec^2} \cos 45°$

$= 16.68 ft/sec^2$

a,y = acceleration due to drag, vertical component (always resists direction arrow is traveling and thus changes from = (-) to (+) during flight.

a,y = a sin A

$= 23.56\ ft/sec^2 2 \sin 45°$

$a,y = 16.68 ft/sec^2$

g,net = gravitational acceleration, net, after being resisted by drag.

g,net = a,y + g

$= (-16.68 - 32.2) \frac{ft}{sec^2}$

$= -48.88 \frac{ft}{sec^2}$

Compute Velocity Components

Vx = velocity, horiz component = V cos A = 200 fps x cos 45° = 141.42 fps.

Vy = velocity, vert component = V sin A = 200 fps x sin 45° = 141.42 fps.

Compute location at end of interval

x = horizontal distance from launch point

$x = x,o + V,x\ TD - \frac{1}{2}ax(TD)^2$

$= 0 + 141.42 \frac{ft}{sec} \times 0.05 sec - 0.5 \times 16.68 \frac{ft}{sec^2} (0.05 sec)^2$

$= 0 + 7.071 ft - 0.021 ft = 7.050 ft$

y = vertical distance from launch point

$y = y,n-1 + V,y\ TD - \frac{1}{2}g,net\ (TD)^2$

$= 0 + 141.42 \frac{ft}{sec} \times 0.05 sec - 0.5 \times 48.88 \frac{ft}{sec^2} (0.05 sec)^2$

$= 0 + 7.071 - 0.061 = 7.010 ft$

Note: Note if elevation difference is plus or minus for future determination of new angle.

Compute Vx at end of interval

$V_x = V_{x,n-1} - a_{,x}\ TD = 141.42 \frac{ft}{sec} - 16.68 \frac{ft}{sec^2} \times 0.05 sec$

$= 141.42 \frac{ft}{sec} - 0.83 \frac{ft}{sec} = 140.59 \frac{ft}{sec}$

Compute Vy at end of interval

$Vy^2 = Vo,y^2 - 2g,net(y,n-y,n-1)$ where

y,n = elevation at end of interval

y,n-1 = elevation at start of interval

$Vy^2 = (141.42ft/sec)^2 - 2 \times 48.88\ ft/sec^2 \times (7.01 - 0)ft = 19{,}315ft^2/sec^2$

$Vy \quad = 138.98\ ft/sec$

Compute new angle

$A = \arctan (Vy/Vx) = \arctan (138.98fps/140.59fps) = \arctan 0.989 = 44.67°$

Refer back to new elevation calculation to ascertain if new angle is plus or minus.

Compute new velocity

$V^2 = Vx^2 + Vy^2 = (140.59ft/sec)^2 + (138.98ft/sec)^2 = 39{,}081\ ft^2/sec^2$

$V \quad = 197.69\ ft/sec$

Compute New Time

Add time difference to total time since launch.

t = 0 + 0.05 sec = 0.05 seconds.

Repeat for next interval

$D = Do(V/Vo)^{1.85} = 386(197.69/200)^{1.85} \quad = 378$ grains

$a = (D/W)g = (378/527) \times 32.2 \quad = 23.08\ ft/sec^2$

$a,x = a \cos A = 23.08 \cos 44.67° \quad = 16.42\ ft/sec^2$

$a,y = a \sin A = 23.08 \sin 44.67° \quad = 16.23\ ft/sec^2$

$g,net = a,y + g = (-16.23) + (-32.2) \quad = -48.43\ ft/sec^2$

$V,x = V \cos A = 197.69 \cos 44.67° \quad = 140.59\ ft/sec$

$V,y = V \sin A = 197.69 \sin 44.67° \quad = 138.98\ ft/sec$

$x = x,n-1 + V,x\ TD - \frac{1}{2}a,x\ (TD)^2 =$

$x = 7.05 + 140.59 \times 0.05 - 0.5 \times 16.42 \times (0.05)^2 \quad = 14.06\ ft$

$y = y,n-1 + Vy\ TD - \frac{1}{2}g,net\ (TD)^2$

$y = 7.01 + 138.98 \times 0.05 - 0.05 \times (-48.43) \times (0.05)^2 = 13.90$ ft

$V_{,x} = V_{,x,n-1} - a_{,x}\ TD = 140.59 - 16.42 \times 0.05 = 139.77$ ft/sec

$V_{,y}^2 = V_{,y,n-1}^2 - 2g_{,net}\ (y_{,n} - y_{,n-1})$

$V_{,y}^2 = 138.98^2 - 2 \times (-48.43) \times (13.90 - 7.01) = 19{,}315 - 667 = 18{,}648$

$V_{,y} = (V_{,y}^2)^{\frac{1}{2}} = (18{,}648)^{\frac{1}{2}} = 136.56$ ft/sec

$A = \arctan(V_{,y}/V_{,x}) = \arctan(136.56/139.77) = \arctan 0.98 = 44.33°$

$V^2 = V_{,x}^2 + V_{,y}^2 = 139.77^2 + 136.56^2 = 19{,}535 + 18{,}648 = 38{,}182$

$V = (V^2)^{\frac{1}{2}} = 195.40$ ft/sec

$t = t_{,n-1} + TD = 0.05$ sec $+ 0.05$ sec $= 0.10$ seconds

Repeat for 3rd interval and so on and so on.

Terminal Velocity

The topic of terminal velocity falls into the category of "trajectories with friction" because an arrow falling straight down at terminal velocity is the final state of an arrow's trajectory if such arrow has been fired from a high cliff. Drag force equals arrow weight when arrow is dropping at terminal velocity. For the sample arrow weighing 527 grains and computed to have a 386 grain drag at 200 ft/sec, terminal velocity is given by the formula:

Eqn (8-3): $V_T = Vo(W/D)^{1/1.85}$

$$V_T = 200 \text{ ft/sec } (527 \text{ gr}/386 \text{ gr})^{1/1.85} = 200 \text{ ft/sec} \times 1.183$$

This arrow, then, if shot straight down off a cliff would pick up only a little speed. An identical looking arrow but one having a thinner wall so as to weigh only 386 grains would neither gain nor lose speed if shot straight down off a cliff. An arrow with flu-flu fletching would slow down.

Computer Program

A print-out of a computer program to perform the calculations follows. This program is for a Hewlett Packard HP41CV equipped with or without printer. The program may be changed to display intermediate results if desired and/or to have the trajectory continue downward after reaching zero elevation. As-is, the computer will (if a printer is attached) compute trajectories starting at whatever launch angle is initially entered and continue until launch angle equals 49°. The final data printed out for each trajectory is the angle at which the arrow hits the ground at zero elevation, the range, the final velocity, the fraction of initial energy remaining, and the time of the flight.

V.O=200 FPS ← LAUNCH VELOCITY, MEDIUM FAST
WT=543 GR ← WEIGHT OF 2117×30 W 125 GR HEAD
T.V.=250 FPS ← TERMINAL VELOCITY IN FREE FALL OF A 2117×30 W (3) 3" FLETCH (APROX.)
DRAG=359 GR ← DRAG (INITIAL) FROM AIR FRICTION
D/W=0.662 ← DRAG-TO-WEIGH RATIO COMPUTED FROM: $\frac{D}{W} = \left(\frac{V_o}{V_t}\right)^{1.85}$

A=39.00DEG
HIT=-48.7DEG
L=276.8 YD
V=138.1 FPS
E=47.7%
T=7.01 SEC

A=40.00DEG ← LAUNCH ANGLE
HIT=-49.8DEG ← HIT-THE-GROUND ANGLE
L=277.8 YD ← RANGE
V=138.3 FPS ← VELOCITY UPON ARRIVAL
E=47.8% ← ARRIVAL ENERGY $= \left(\frac{V_F}{V_o}\right)^2 \times 100$
T=7.16 SEC ← DURATION OF FLIGHT

A=41.00DEG ← LAUNCH ANGLE FOR MAX. RANGE
HIT=-50.9DEG
L=277.9 YD ← MAXIMUM RANGE
V=138.5 FPS
E=48.0%
T=7.29 SEC

A=42.00DEG
HIT=-51.9DEG
L=277.8 YD
V=138.8 FPS
E=48.2%
T=7.42 SEC

A=43.00DEG
HIT=-53.0DEG
L=277.5 YD
V=139.1 FPS
E=48.4%
T=7.55 SEC

A=44.00DEG
HIT=-54.0DEG
L=277.0 YD
V=139.5 FPS
E=48.6%
T=7.68 SEC

A=45.00DEG
HIT=-55.0DEG
L=276.1 YD
V=139.8 FPS
E=48.9%
T=7.81 SEC

Figure 8-2

COMPUTER MODEL FOR FLIGHT SHOOTING AN ARROW FROM LAUNCH ANGLES OF 39° THROUGH 45°

Thomas L. Liston, P.E.
Mechanical Engineer
June 1987

```
 01♦LBL "DRAG"
 .05 STO 05 0 STO 06
 STO 07 STO 10 1
 STO 13 FS? 55 SF 21
 SF 12

 13♦LBL 00
 FIX 0 RCL 00 CLA
 ARCL 00 "⊦ FPS?"
 PROMPT STO 00 CLA
 ARCL X "⊦ FPS" AVIEW
 STO 09

 26♦LBL 01
 RCL 01 CLA ARCL 01
 "⊦ DEG?" PROMPT STO 01
 STO 08

 34♦LBL 02
 RCL 02 CLA "WT="
 ARCL 02 "⊦ GR?" PROMPT
 STO 02 CLA "WT="
 ARCL X "⊦ GR" AVIEW

 47♦LBL 03
 RCL 03 CLA "DRAG="
 ARCL 03 "⊦ GR?" PROMPT
 STO 03 CLA "DRAG="
 ARCL X "⊦ GR" AVIEW
 RCL 02 / FIX 3 "D/W="
 ARCL X AVIEW

 66♦LBL 04
 67♦LBL 05
 RCL 02 RCL 03 / 1.85
 1/X Y↑X RCL 00 *
 FIX 0 CLA "T.V.="
 ARCL X "⊦ FPS" AVIEW
 ADV CF 21

 84♦LBL 06
 CF 21 "DRAG" RCL 09
 RCL 00 / 1.85 Y↑X
 RCL 03 * STO 11

 95♦LBL 07
 "ACCELERATIONS" RCL 11
 RCL 02 / 32.2 *
 STO 12 RCL 08 COS *
 STO 14 RCL 12 RCL 08
 SIN * CHS -32.2 +
 STO 15

 115♦LBL 08
 "V,X V,Y" RCL 09
 RCL 08 COS * STO 16
 RCL 09 RCL 08 SIN *
 STO 17

 127♦LBL 09
 "X" RCL 05 2 /
 RCL 14 * CHS RCL 16
 + RCL 05 * ST+ 07

 140♦LBL 10
 CLA "Y" RCL 05 2 /
 RCL 15 * RCL 17 +
 RCL 05 * STO 18
 ST+ 06 X<=0? XEQ 20
 FIX 1 "HT=" ARCL 06
 AVIEW

 160♦LBL 11
 "V,X" RCL 05 RCL 14 *
 ST- 16

 166♦LBL 12
 "V,Y" RCL 18 RCL 15 *
 2 * RCL 17 X↑2 +
 SQRT STO 17

 178♦LBL 13
 "A" RCL 17 RCL 16 /
 ATAN RCL 13 * STO 08
 FIX 2 FC? 55 VIEW X

 190♦LBL 14
 "V" RCL 16 X↑2 RCL 17
 X↑2 + SQRT STO 09

 199♦LBL 15
 "T" RCL 05 ST+ 10

 203♦LBL 16
 "REPEAT" RCL 06 X<=0?
 GTO 21 RCL 06 STO 19
 RCL 07 STO 20 RCL 08
 STO 21 RCL 09 STO 22
 RCL 10 STO 23 GTO 06

 219♦LBL 20
 -1 STO 13 RTN

 223♦LBL 21
 "HORIZ" RCL 19 RCL 06
 - RCL 19 / 1/X
 STO 24 RCL 07 RCL 20
 - RCL 24 * RCL 20 +
 STO 25 RCL 08 RCL 21
 - RCL 24 * RCL 21 +
 STO 26 RCL 09 RCL 22
 - RCL 24 * RCL 22 +
 STO 27 X↑2 RCL 00 X↑2
 / 100 * STO 28
 RCL 10 RCL 23 -
 RCL 24 * RCL 23 +
 STO 29

 271♦LBL 22
 FS? 55 SF 21 BEEP
 FIX 2 CLA "A="
 ARCL 01 "⊦DEG" AVIEW
 FC? 55 STOP CLA FIX 1
 "HIT=" ARCL 26 "⊦DEG"
 AVIEW FC? 55 STOP
 FIX 0 CLA "L="
 ARCL 25 "⊦ FT" AVIEW
 FC? 55 STOP CLA "V="
 ARCL 27 "⊦ FPS" AVIEW
 FC? 55 STOP CLA "E="
 ARCL 28 "⊦%" AVIEW
 FC? 55 STOP FIX 3 CLA
 "T=" ARCL 29 "⊦ SEC"
 AVIEW FC? 55 STOP ADV

 322♦LBL 23
 1 ST+ 01 0 STO 06
 STO 07 STO 10 1
 STO 13 RCL 00 STO 09
 RCL 01 STO 08 49
 RCL 01 X>Y? BEEP X>Y?
 STOP GTO 06 END
```

```
LBL"DRAG
END
616 BYTES
.END.
```

```
173 FPS
WT=549 GR
DRAG=110 GR
D/W=0.200
T.V.=413 FPS
```

173 FPS
WT=549 GR
DRAG=110 GR
D/W=0.200
T.V.=413 FPS

PRREG

R00=	173.00	= V_o = Velocity @ launch (ft/sec)
R01=	41.00	= A_o = Angle of launch (degrees)
R02=	549.00	= W = Weight of arrow (grains)
R03=	109.80	= D_o = Drag @ launch velocity (grains)
R04=	150.00	(not used)
R05=	0.05	= Δt = Time interval (seconds)
R06=	27.24	= Y = Elevation (feet)
R07=	32.49	= X = Horizontal position (feet)
R08=	38.91	= A = Angle of flight (degrees)
R09=	166.27	= V = Velocity (ft/sec)
R10=	0.25	= T = Time since launch (sec)
R11=	102.02	= D = Drag (grains)
R12=	6.07	= a = Acceleration due drag (ft/sec²)
R13=	1.00	= (+/-) of A ("up" if +; "down" if −)
R14=	4.70	= a_x = Acceleration in x-direction
R15=	-36.05	= g_{net} = Net vertical acceleration
R16=	129.37	= V_x
R17=	104.44	= V_y
R18=	5.27	= Δy
R19=	27.24	= Y, previous
R20=	32.49	= X, "
R21=	38.91	= A, "
R22=	166.27	= V, "
R23=	0.25	= T, "
R24=	0.12	= F, interpolation
R25=	64.76	= X, final
R26=	-63.16	= A, "
R27=	24.48	= V, "
R28=	1.33	= E, "
R29=	1.71	= T, final

ARROW STRUCTURAL STRENGTH

Shaft Strength:

When an arrow is shot, the arrow accelerates in three directions: forward, left and down. Why left and down? The answer to that question is covered in the chapter on arrow vibrations. For now, let us examine the acceleration forces involved in the straight-ahead direction.

Forward Acceleration Forces:

When an arrow is fired, tremendous accelerations are involved. These forces verge on destroying the arrow. Just how strong are these forces? It depends. One might expect the forces would be the same as measured on the bow's force-draw curve. Long bows and/or recurve bows develope maximum pull when drawn fully. Were one to shoot a very heavy arrow (a crowbar on roller skates, for instance) with a 60# recurve, the initial push would be 60#. As the crowbar began moving, the force of the bow string pushing against it would diminish in the same way the forces diminish as an archer "lets down" his bow without firing it.

The "g" forces involved would depend upon the weight of the crowbar. If it weighed 60# and were fired from a 60# bow, the initial "g" force would be 1.0-g. If the crowbar weighed 30#, it would accelerate twice as fast, or 2-g's. Were it a real arrow weighing 539 grains or 0.0770#, the acceleration would tend toward but not reach 60#/0.0770# = 779-g's! The initial force addressed to the nock of an arrow is almost equal to the bow's static force at full draw if the arrow's weight greatly exceeds the bow's virtual mass. If the bow's virtual mass equaled the arrow's weight (on the other end of the spectrum), the initial force would be 50% of full draw weight.

The initial load imposed on the nock of an arrow, then, approaches the load that would be imposed if the arrow were used to hold the bow at full draw. You might try putting a strung arrow against a block and then pushing the bow. You might, but you had better not: The arrow will break, probably causing injury and damage. The author tried nocking an arrow and gradually pushing against a block. When the bow was about half way drawn it became obvious that the arrow was becoming unstable and would break if the experiment were continued. Why, then, doesn't the arrow break when fired? Because the load carried by the shaft is 60# at the nock but diminishes along the arrow's length, becoming zero at the very tip. Why does the force diminish along the arrow's length? Because:

Force = mass x acceleration.

The acceleration is the same along the entire length of the arrow. The mass ahead of the force is less and less along the length, however, until there is no more mass left at the very tip.

In the following figure, we will take an actual arrow's weight distribution, as described in Easton Aluminum Hunting Shaft Selection Chart, as follows:

Broadhead	125 grains
Insert	30 grains
2117 shaft, 29" long ...	349 grains
Nock & fletching	35 grains

Total = ... 539 grains = 0.0770 pounds.

A 60# recurve bow would probably address a peak "push" of about 48# at the arrow's nock. Five inches down the shaft, the nock, fletching & 5/29.75ths of the shaft are behind. The mass ahead has been reduced to 445 grains, and the accelerating force has been reduced to 48# x (445/539) = 40#. At the base of the broadhead the mass ahead has been reduced to that of the broadhead alone, or 125 grains. The propelling force here is 48# x (125/539) = 11#.

<u>Buckling of Column Under Own Weight</u>

The analysis of strength needed to tolerate huge accelerations would be the same as the analysis of a column's buckling under it's own weight if the arrow had no tip. R. Frisch-Fay, Lecturer in Civil Engineering at the University of New South Wales in his book, "Flexible Bars" published by Butterworths, Washington, 1962, does devise a formula, which is:

$$W_{,critical} = 7.84\ EI/L^2$$

= critical <u>shaft</u> weight.

When fired from a bow whose string delivers a thrust of 48 pounds, the arrow actually "weighs" 48 pounds.

<u>Slender Columns</u>

The arrow tip's weight is transmitted through the entire length of the arrow during acceleration. Thus, the ability of the shaft to accelerate that tip is comparable to the ability of a slender column to carry a static load. The formula for the critical load which can be carried by a slender column without buckling is:

$$W_{,critical} = pi^2EI/L^2 =$$

critical <u>tip</u> weight

Structural engineers developed the phrase "slender column" to predict when a column needs lateral bracing to prevent buckling. An arrow being fired is in a similar but different situation. It is similar because the arrow is subjected to forces which will definitely cause it to buckle. It is different for two reasons. One is that the launch is completed before the arrow has a chance to buckle to the point of permanent deformation. The second difference is that the forces being carried by the arrow's shaft vary from tip to nock, whereas a structural column carries the same load throughout it's length. The load carried by an arrow shaft is maximum (about 48# when fired from an 60# bow) at the nock and diminishes to zero at the arrow's tip. Flight shooters who compete for maximum distance use barrel shaped arrows, presumably for the reasons given above.

Both the "slender column" effect and the "buckling under own weight" effect are simultaneously at work. How the two are combined to compute needed arrow strength is quite beyond this mechanical engineer, however. I hereby challenge any interested engineer to develope the structural analysis needed to predict arrow failure. Some points are obvious without resorting to math. One point is that the need for maximum strength will occur somewhere between the nock and the midpoint. Why? Well, were a constant force carried, the midpoint would need to be strongest. That is intuitively obvious. In the case of the arrow, since forces are greatest on the nock side, need for peak strength will be short of midway. "Strength" refers to the ability to resist bending, not crushing, as the ability of a section of shaft to resist simple in-line crushing is massive and need not be analyzed.

Sample Calculation of Structural Strength

This calculation will be divided into two parts. The first part will use the "slender column" concept to evaluate the ability of the arrow shaft to propel the arrow tip, consisting of insert plus broadhead. The answer sought will be, "What fraction of the arrow's strength is required just to accelerate the tip and insert?" The second part will ignore the tip and use the "buckling under own weight" concept. The answer sought will be, "What fraction of the arrow's strength is required just to support the arrow shaft's weight?"

Arrow Data:

Tip (broadhead)	=	155 gr
Shaft, 29" long, 2117,	=	349 gr
Nock & fletching	=	35 gr
Total Weight	=	539 gr
Diameter of shaft	=	21/64"
Wall thickness	=	0.017"
Aluminum's UTS	=	88,000 psi
Aluminum's density	=	174 lb/ft^3
Elastic moduli in tension & compression	=	10.5×10^6 psi
Elastic moduli in shear	=	4.0×10^6 psi

Maximum acceleration force = 48 lb.

Slender Column's Ability to Accelerate Tip + Insert

Acceleration, max = Fmax/m
= 48#/(539 gr/7,000 gr/#)
= 48#/0.0770#
= 623 g's

Weight of tip + insert during maximum acceleration =

155 gr x 623-g's x 1#/7,000 gr
= 13.8 lb.

Critical load, tip + insert
= $(Pi)^2EI/L^2$

$E = 10.5 \times 10^6 \#/in^2$
= elastic modulus of aluminum
L = 29" = length of shaft
I = moment of inertia, hollow shaft
$I = (Pi/64)/(D^4 - d^4)$ where
D = outside diameter and
d = inside diameter.
d = D - 2t = 21/64" - 2 x 0.017"
= 0.3281" - 0.0340"
= 0.2941"
$I = (3.1416/64)(0.3281^4 - 0.2941^4)$
I = 0.04909 (0.01159 - 0.00748)
= 0.04919 x 0.0041 = 0.000202 in^4
Critical tip load = $W_{,critical}$
= $(Pi)^2EI/L^2$ =

$$\frac{(3.14)^2 \times 10.5 \times 10^6 \#/in^2 \times 0.000202 in^4}{(29\ in)^2}$$

= $9.87 \times 10.5 \times 2.02 \times 10^2 \# / 841$
= 24.9#.
Percent of strength
= (load/strength) x 100
= 13.8#/24.9# = 55%.

Buckling Under Own Weight

The weight of nock and fletching can be ignored, mainly because they are near the base where their weight is most easily carried. The weight of the tip has to be ignored because the math to cover it is not known. Rather, it was calculated separately in the previous paragraph. Thus, only the shafts ability to carry its own weight is being considered. The formula is:

$$W_{critical} = 7.84\ EI/L^2 = \frac{(7.84 \times 10.5 \times 10^6 \#/in^2 \times 0.000218 in^4)}{(28\ in)^2}$$

$$W_{critical} = 19.75\#$$

Actual weight during 623-g acceleration of shaft weighing 349 grains is:

W_a = 349 gr x 623-g/(7000 gr/#) = 31.1#.

Percent-of-strength = $W_{actual}/W_{critical}$ = 31.1#/19.75# = 1.57 = 157%.

Conclusion: This arrow is in the act of buckling under its own weight when launch is initiated.

Comparison of Cataloged Recommendations with Calculated Values:

Easton Aluminum does a fine job of cataloging their arrow data and recommendations. Just for the fun of it, I've calculated the data for their recommended use of 2117 hunting arrows. The arrow diameter and wall thickness is the same in all cases. The arrow lengths, weights, recommended bow peak draw forces differ. I've assumed that all bows are 80% efficient, delivering a peak accelerating force equal to 80% of peak draw force. The recommendations for compound bows having 50% let-offs are used.

Data:

Broadheads weigh 125 grains & inserts weigh 30, for a total of 155 grains.

E = 10,500,000#/in^2 for aluminum.

I = (pi/64) x (D^4 - d^4) = 0.0002018in^4.

Critical tip weight = pi^2 EI/L^2 = 20,914/L^2.

$$G's = \frac{(0.8 \times \text{peak force})}{(\text{grains}/7000)}$$

Critical weight for buckling ignores broadhead, tip, nock and fletching weights. Thus buckling weight = total - 125 - 30 - 35 grains.

Table 9-1 ... 2117 ARROWS' STRUCTURAL LOADS @ RECOMMENDED LENGTHS & BOW WEIGHTS

Length (in)	Arrow weight (gr)	Peak draw force (lbs)	Slender column strength (lbs)	Peak Acceleration (G's)	Tip Weight (lbs)	% of column strength (%)	Actual shaft weight (gr)	Accelerated shaft weight (lbs)	Critical shaft strength (lb)	% of buckling strength (%)
27	515	80.5	28.7	875	19.4	68	325	40.6	22.8	178
28	527	74.5	26.7	792	17.5	66	337	38.1	21.2	180
29	539	68.5	24.9	712	15.8	63	349	35.4	19.8	179
30	551	62.5	23.2	635	14.1	61	361	32.8	18.5	177
31	563	56.5	21.8	562	12.4	57	373	29.9	17.3	173
32	575	50.5	20.4	492	10.9	53	385	27.1	16.2	167
33	587	44.5	19.2	424	9.4	49	397	24.0	15.3	157

The above table indicates that the arrow could be expected to buckle even if it had no tip. The calculations say that a nearly constant fraction of buckling strength is used at all recommended selections, varying from 180% to 157%.

A better theoretician could doubtlessly integrate the formulas for strength requirements to avoid buckling under own weight plus to avoid buckling when carrying the weight of the tip. The bigger question, too, of how to predict what conditions will actually destroy the arrow would be the next step. Finally, it would be very nice to know which arrow should shoot the most accurately and why.

An arrow selected according to Easton's recommendations is in the act of buckling when being fired. The art of arrow selection seems to be to match arrow to bow in such a way that the arrow buckles neither too much nor too little.

In the great little book, "Archery, The Technical Side" it was reported that arrows fired straight ahead did not need to be matched to their bows! When the arrows were restrained left/right & up/down so as to center fire and when they were shot by experts using pinch type releases, arrows of vastly different spine all fired straight ahead. Arrows which were much too flimsy simply collapsed.

Flight Shooters

Flight shooters who compete for maximum distance use barrel shaped arrows, presumably for the reasons given above. The record for a hand-held bow shot is over 1000 yards! The formula for the minimum velocity needed to shoot an arrow is: V = square root of (R x g). Solving that for 1000 yards gives:

$$V = (1000 \text{ yds} \times 3 \text{ ft/yd} \times 32.2 \text{ ft/sec2})^{\frac{1}{2}} = 311 \text{ ft/sec!}$$

However, in "Saracen Archery" the authors explain that Turkish flight arrows were balanced to achieve an angle-of-attack to the air which resulted in a gliding effect. This effect means that all bets are off when compared to ballistic trajectories. A glider aircraft can obviously go much farther with a given initial speed than could an aircraft in a ballistic trajectory. The same is true of arrows.

Applicability

"Slender column" analysis applies to arrow shafts propelling heavy tip loads, such as hollow aluminum shafts having having inserts and heavy tips. Weight of shaft is ignored.

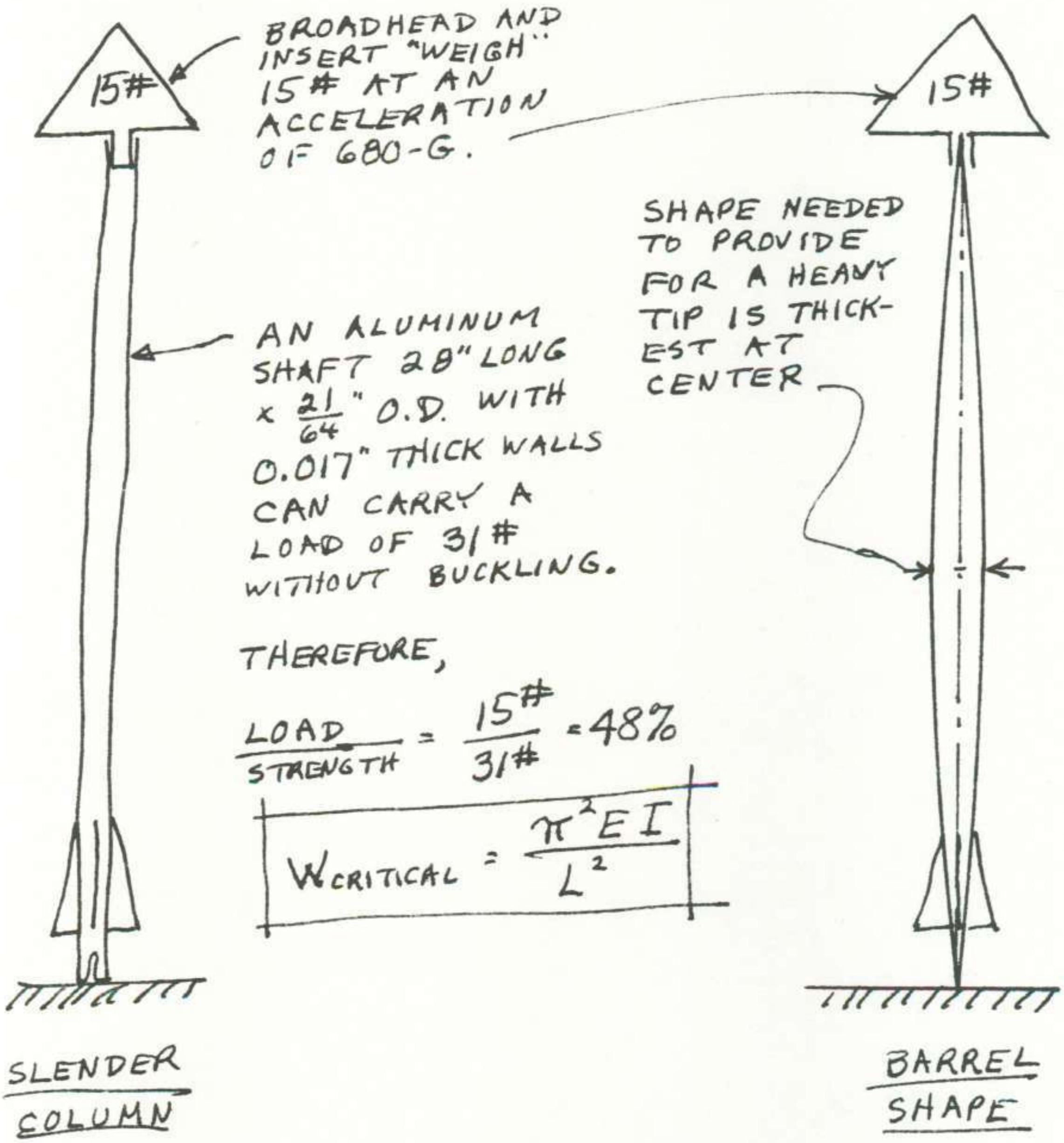


Figure 9-1 ... "SLENDER COLUMN" ILLUSTRATION

Applicability

An arrow without a tip buckles under it's own "weight" if $W \geqq 7.84EI/L^2$. (With a tip, it will buckle earlier).

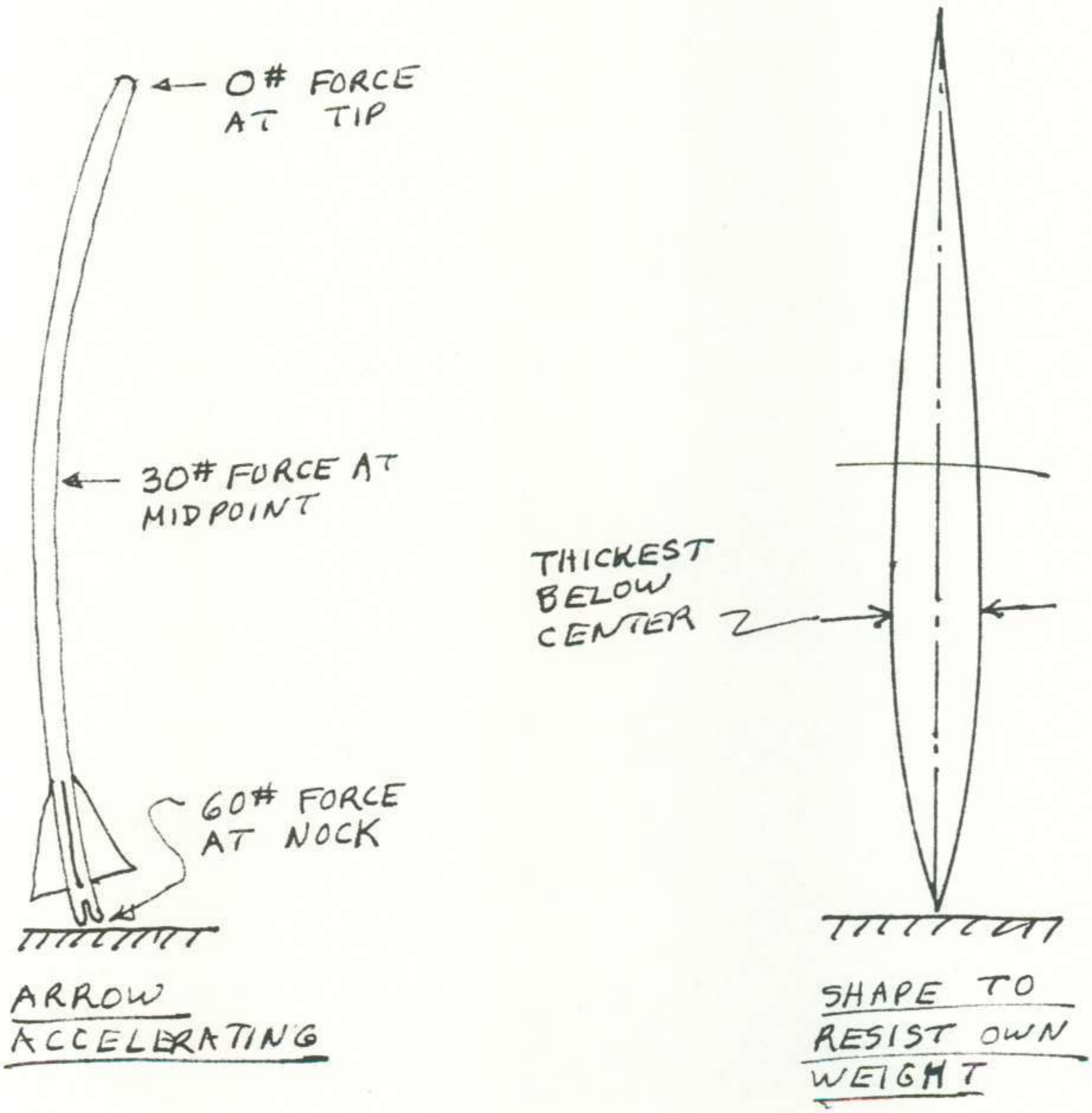


Figure 9-2 ... BUCKLING UNDER OWN WEIGHT

ARROW VIBRATION

Arrow spine

In my youth, I heard that arrows must be selected to "wrap around" the bow. In those days of long bows, it seemed logical. The string would draw the arrow toward the center of the bow, which was in the way of the arrow. Thus the arrow needed to somehow get around the bow. It was explained to me that if the arrow were too flexible it would wrap too far around. If it were too stiff, it would be deflected by the bow handle.

I was away from archery for about 30 years. During that time they developed bows with handles offset to get out of the arrow's way. I was surprised, then, to hear that the same old matching of the arrow to the bow was still required. Why? I could understand that the arrow needed to be strong enough to tolerate the bow's acceleration forces. But I couldn't understand why there was such a thing as "too stiff" an arrow. It seemed to me that since the arrow didn't need to wrap around anything, the stiffest arrow would be the best arrow.

Were the arrow shot exactly straight ahead without any guidance, which way would it go? The question is similar to asking which way will a broom handle tend to fall if you try to balance it in the palm of your hand. It could go most any direction. Thus the need to guide the arrow during launch is evident. If the arrow were restrained from going either left or right or up or down, my initial feeling that the stiffest arrow would be the best would probably be correct. The only direction which the arrow can be guided is to the left. The bow is in the way of the arrow's travel to the right. So there is no direction to send it other than to the left. This being the case, we are back to the days of the long bow: The arrow has to be pushed left & therefore the stiffness of the arrow has got to be matched to the bow.

A similar consideration applies to the up-down guidance of the arrow rest. When the arrow is accelerating at about 300-g, The downward pull of 1 g is truly negligible. That must be why the recommendation is that the arrow nocking point be placed about 3/8" above the arrow rest. This insures that the arrow's tip will be accelerated toward the arrow rest, which is downward, during launch. Why should downward be better than upward? No important reason, except that since the arrow must rest while waiting to be launched, why not let that same arrow rest be the vertical restraint?

Back to the analogy of the balanced broom handle: The equivalent is that of holding the broom handle almost but not quite straight up, using the help of corner walls to keep it from falling in any direction.

An arrow's "spine" is a very complicated thing. The author has not found in the literature an engineering explanation of satisfactory accuracy. The author leaves to some interested archer/engineer/mathematician the challenge of describing mathematically how an arrow behaves during launch. It is a complicated topic. The best writings on the topic found by this author are found in a book entitled "Archery: The Technical Side" by Hickman, Nagler & Klopsteg published in 1947 by The North American Press, Milwaukee, Wisconsin. Tom Hughes of Ranging, Incorporated of East Bloomfield, New

York, told me about this great little book. The librarian at the West Valley Branch of the San Jose Library did what I considered an amazing job of locating the book in the Putnam Library of Palomar College, San Marcos, California.

One <u>very</u> interesting thing reported in that book is that a center-fired, fully restrained arrow released from a pinch-type release did indeed shoot straight ahead, just like one would expect. Arrows of various "spine" all shot the same! The same bows and arrows shot with large errors when released with tabs or bare fingers, however.

A couple of other interesting items reported therein about the dynamics of arrow launching: A properly matched arrow is in firm contact with the arrow rest during the first few inches of travel; thereafter, the arrow does not touch the bow at all!

Arrow Vibration

Arrows in flight vibrate. The points about which they vibrate are called nodes. Nodes are located roughly at quarter points.

Vibration During Launch

The arrow's initial vibration has both nock and tip moving left. Why left? The arrow <u>must</u> vibrate in some direction because it is in the act of buckling. The arrow rest, being left of center, starts the tip going left. The fingers give the nock a bit of left motion when releasing. Since the arrow is going to buckle anyway, the direction is initiated by the fingers and by the arrow rest.

The speed of the arrow's vibration depends upon the arrow's natural frequency. The displacement depends upon the strength, inertia and acceleration.

Bow's Natural Frequency

I strongly suspect that the natural frequency of the arrow's vibration and the natural frequency of bow need to have some logical relationship to one another. By "natural frequency" I refer, in the case of an arrow, to the rate at which a bent arrow vibrates when the bending force is released. In the case of the bow, "natural frequency" strictly speaking refers to a "dry fire" scenario. From the moment the string is released until the moment the arrow separates, the bow is behaving like a vibrating object. The differences are several. First, of course, is that the bow only goes through what amounts to 1/4th of a full cycle of vibration. Mathematically, this poses no problem. Second, "dry fire" is not the usual. Thus the bow's "natural" frequency is perhaps the wrong frequency to be analyzing. Perhaps it should be the equivalent frequency of the arrow-bow combination. I leave that to the mathematician who tackles the problem.

Analysis of the bow's frequency, whether dry fired or when firing an arrow, was covered in the discussion of virtual mass elsewhere in this book.

Arrow's Natural Frequency

Analysis of the arrow's natural frequency is tougher. The several mathematical approaches are:

1. Assume the arrow is a shaft of uniform weight, strength and cross section from nock to tip. Assume (as is true) that vibration will take place about two nodal points.

2. Having computed the vibration math for a uniform shaft, complicate the math by assuming that

equal weights are added to the tip and to the nock end.

3. Having figured that out, complicate it further by accounting for the fact that the weight of the tip is different than that of the fletching and nock.

4. Take the next step, which is to account for the fact that the weight on the tip is not only different from that on the nock but it's distance from the center of the shaft is also different.

5. Having figured out exactly how and why an arrow vibrates, compute the damping features of it's fletching.

6. Having figured out all about the vibration, decide what practical lessons can be learned. In other words, "So what?".

Arrow Design

I suspect that a number of possible lessons would be learned. One might conclude that there is an advantage to having equal weights on tip and tail. Even more interesting would be deciding that some other weight distribution would be best.

In looking for answers to the question of "So what?", you have to keep in mind the objectives of various archers.

Flight Shooters' Objectives:

Flight shooters want maximum range. For them, the arrow's shape should be (and is) barrel-shaped with the maximum strength somewhat aft of center so as to accept maximum acceleration without structural failure. Such an arrow shape has it's center of gravity aft of center. Once launched, the flight shooter wants an arrow that has the least possible friction. Since fletching causes major friction, the flight shooter is most anxious to have minimum fletching. An arrow with center of gravity forward of center should need no fletching. Yet the flight shooter's arrow needs maximum strength aft of center. Where is the best compromise? Perhaps a spring-loaded weight inside the arrow shaft would be a huge help, with the weight far aft during launch and springing full forward immediately after launch. Such an arrow would need no fletching, could accelerate huge weights with minimum shaft strength, and could hide the feature entirely! (Should I patent the idea?) In "Saracen Archery" is described the weighting of flight arrows so as to not tip forward quite as fast as the trajectory, resulting in a gliding effect. The record for a hand-held bow shot is over 1000 yards! The formula for the minimum velocity needed to shoot an arrow is:

$V = (R \times g)^{\frac{1}{2}}$. Solving that for 1000 yards gives:

$$V = \left(1000\,yd \times 3\,\frac{ft}{yd} \times 32.2\,\frac{ft}{sec^2}\right)^{\frac{1}{2}}$$

$$= 311 \text{ ft/sec!}$$

However, in "Saracen Archery" the authors explain that Turkish flight arrows were balanced to achieve an angle-of-attack to the air which resulted in a gliding effect. This effect means that all bets are off when compared to ballistic trajectories. A glider aircraft can obviously go much farther with a given initial speed than could an aircraft in a ballistic trajectory. The same is true of arrows.

Target Shooters Objectives:

Target shooters want accuracy. Each arrow should shoot exactly the same as all others. Theoretically, if each arrow were identical to the others, spine would make no difference. To some extend, with today's aluminum arrows, it is a fact that too-stiff arrows will shoot fairly accurately in spite of not being well matched to the bow. Arrows which are too stiff or not stiff enough leave the bow going somewhat sideways. It is then up to the fletching to straighten them out. It is presumably because the fletching's interaction with the air is less than identical in each case that mis-matched arrows do not "group" as well. Well matched arrows will leave the bow traveling straight. Thus the only work needed of the fletching is to make them continue going straight and to tip them downward to match their trajectory..

Two forces tend to make the arrow not continue going straight. One is gravity. Were an arrow shot on the moon where there is no atmosphere, it would arrive at the target with the same up-aiming tilt as it had at the moment of launch. True, it would actually be traveling downward (same as on earth) at the moment of impact, but it would be pointing upward, the same as it was at the moment of launch. Here on earth we need fletching to make the arrow rotate so as to stay aligned with the air stream as the arrow's path changes from upward to downward. On the moon, the arrows would actually hit targets going somewhat sideways, resulting in damage to the arrows.

The other force tending to make a straight-flying arrow turn is the weather-cocking tendency. Were a clean target arrow's center of gravity forward of center, the arrow's after body would act like a weather cock, tending to keep the arrow flying straight. Such an arrow would need minimum fletching. If the center of gravity were aft, however, the arrow's tendency would be to turn around and fly backwards.

<u>Bowhunters' Objectives</u>:

Bowhunters want accuracy (same as target shooters) but also want speed and energy. They need speed to minimize the consequences of errors in estimated range. They need energy for good arrow penetration. Broadheads act like fletching at the wrong end of the arrow. Broadheads try to turn the arrow around. To offset the broadhead's forces, hunting arrows need more fletching than other arrows. The broadhead's weight changes the arrow's natural frequency. On the plus side, the broadhead's weight tends to put the arrow's center of gravity forward, thus stabilizing it's flight.

Range Estimation Error Consequences

A bowhunter's ability to accurately estimate range is as important or, in some cases, more important than his ability to launch an arrow accurately. Think about it for a moment. A hunter capable of putting an arrow within a few inches of the intended point at fifty yards will miss by a wide margin if the target is actually 40 or 60 yards away. How far will he miss by? It depends upon three factors: the range, the arrow's velocity, and whether he estimates long or short.

Miss distance due to range estimate error:

For an archer who has pin sights, it is a simple matter to ascertain how far the miss will be by simply looking at his pins. For instance, say the target is actually 50 yards away. Stand 50 yards from a target. Put 40-yard pin on target. With 40-yard pin on target, look at where the 50-yard pin in pointed, because that is where the arrow will hit if you launch! Repeat at any distances of interest. See figure 11-1.

Mathematically, error at closest-point-of-approach is equal to the mistake in range multiplied by the sine of the arrow's launch elevation. Which launch elevation to use depends upon whether mistake is short or long. The formulas are:

$M_{high} = E_R \text{ sine } A_{R=long}$
Eqn #11-1, where

M_{high} = miss distance when estimated range is longer than actual.
E_R = error in range estimate
$A_{R=long}$ = Angle of launch to hit target at estimated range.

$M_{low} = E_R \text{ sine } A_{R=R}$,
Eqn #11-2, where

M_{low} = miss distance when estimated range is shorter than actual.
E_R = error in range estimate
$A_{R=R}$ = Angle of launch to hit target at actual range.

A 10% farther range estimation error will result in a greater miss than will a 10% shorter range estimation. On average, then, a hunter is more apt to shoot high than he is to shoot low. The difference, however, is reasonably small and thus it is convenient and reasonably accurate to use Equation #11-2 for both high and low calculations. Table 11-1 illustrates the influence of range estimation errors with three different bows and at ranges from 10 to 90 yards.

The data, when plotted, shows that the amount by which an arrow will miss for a given percent error in estimation of range is a square function of the range. Doubling the range will quadruple the miss. It also shows the benefit of high speed arrows. For instance, assuming an error in range estimation of 10% of whatever the range is, and assuming that a 12" error can be tolerated, the following is true:

A 42# bow, 140 fps arrow, can be shot at 37 yards.

A 58# bow, 160 fps arrow, can be shot at 42 yards.

An 84# bow, 240 fps arrow, can be shot at 63 yards.

Plotting that data shows that the distance from which an arrow can be shot for a given acceptable error is linear with speed. Doubling arrow

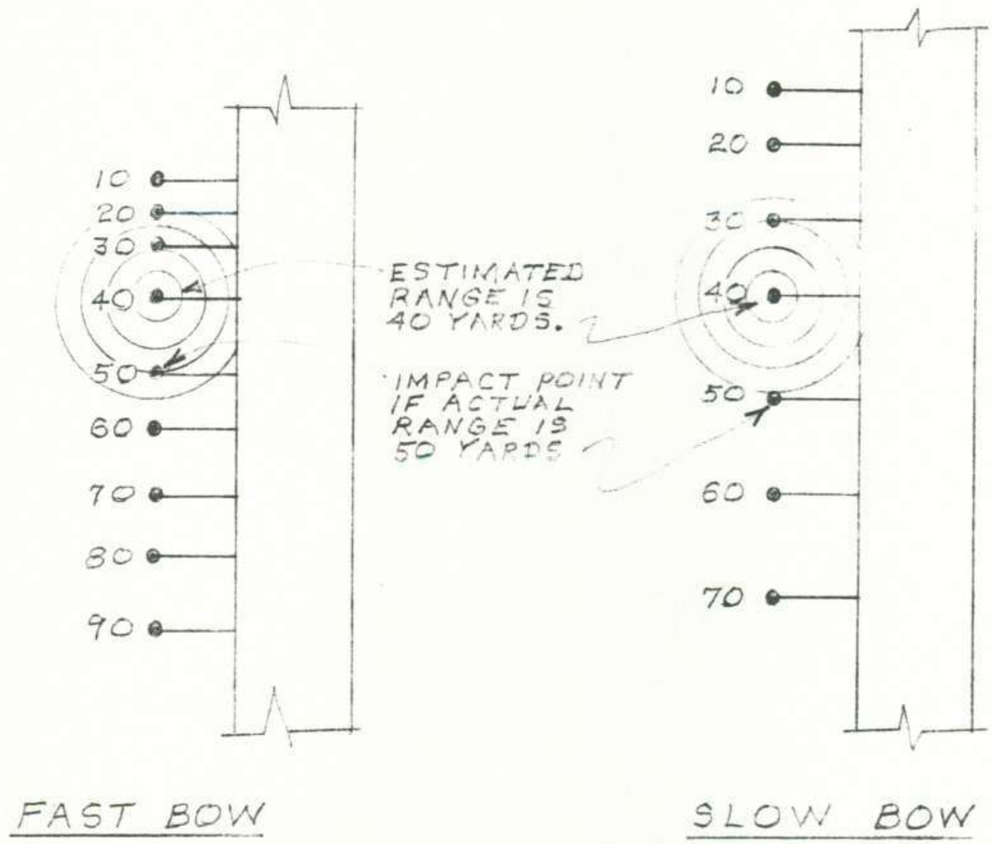


Figure 11-1 ... PIN SIGHT PICTURES

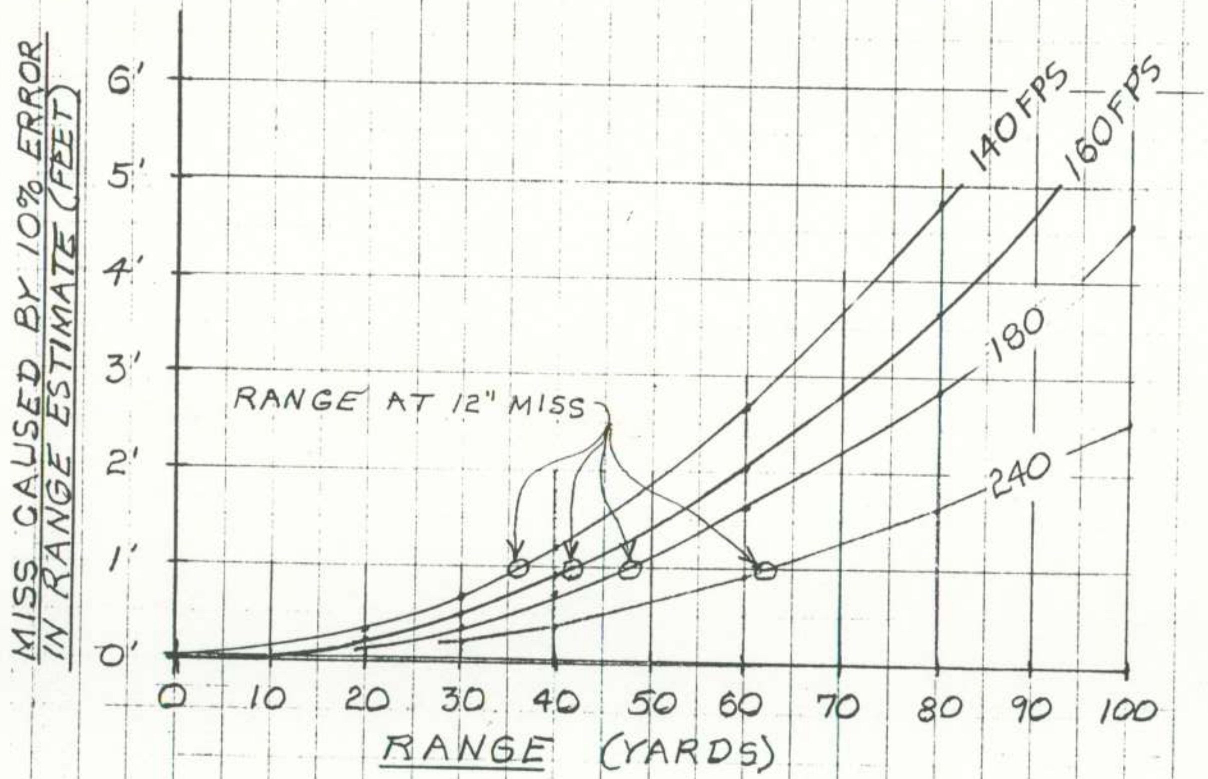
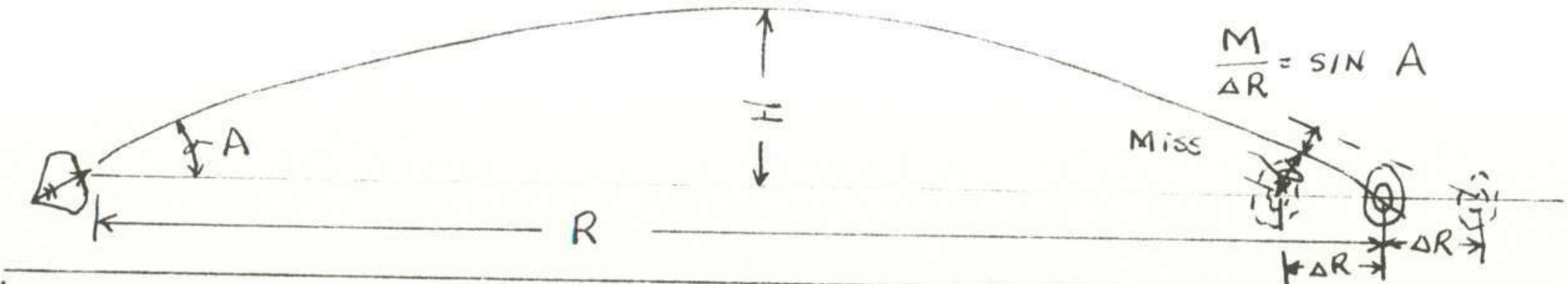


Figure 11-2 ... MISSES vs RANGE vs SPEED



TRAJECTORY DATA

V	140 ft/sec			160 ft/sec			180 ft/sec			240 ft/sec		
Time	8.7 sec			9.9 sec			11.2 sec			14.9 sec		
R,max	203 yds			265 yds			335 yds			596 yds		
H,max	609 ft			795 ft			1,006 ft			1,789 ft		
Range (yds)	Angle (deg)	H (ft)	10%Miss (ft)	Angle (deg)	H (ft)	10%Miss (ft)	Angle (deg)	H (ft)	10%Miss (ft)	Angle (deg)	H (ft)	10%Miss (ft)
0	0	.0	.00	0	.0	0	0	.0	0	.00	.0	0
10	1.4	.4	.07	1.1	.3	.06	.9	.2	.04	.48	.1	.03
20	2.8	1.5	.30	2.2	1.1	.23	1.7	.895	.18	.96	.5	.10
30	4.3	3.3	.67	3.2	2.6	.51	2.6	2.0	.40	1.44	1.1	.23
40	5.7	6.0	1.19	4.3	4.6	.91	3.4	3.6	.72	1.92	2.0	.40
60	8.6	13.6	2.69	6.5	10.3	2.05	5.2	8.1	1.62	2.89	4.5	.91
80	11.6	24.7	4.83	8.8	18.5	3.67	6.9	14.5	2.88	3.9	8.1	1.61
100	14.8	39.5	7.65	11.1	29.4	5.77	8.7	22.9	4.52	4.8	12.7	2.52
203	45.0	304.3	43.04	25.0	141.8	25.71	18.6	102.5	19.43	9.9	53.4	10.51
265				45.0	397.5	56.22	26.1	194.7	34.97	13.2	93.2	18.15
335							45.0	503.1	71.15	17.1	154.9	29.61
596										45.0	894.4	126.49

"V" = initial arrow velocity. $g = 32.2\ ft/sec^2$
"Time" = elapsed time for arrow shot straight up to make round trip. $= 2V/g$
"R,max" = maximum range for arrow shot at 45°. $= V^2/g$
"H,max" = maximum height for arrow shot straight up. $= V^2/2g$
"Angle" = arrow elevation at launch. $= 0.5 \arcsin(RG/V^2)$
"H" = maximum altitude of arrow during flight. $= V^2 \sin^2 A / 2g$
"10% Miss" = how far arrow will miss target if range is wrong by 10%. $= \frac{R}{10} \sin A$

Table 11-1 ... TRAJECTORY DATA

Thomas L. Liston, P.E.
May 11, 1985

speed allows shooting from twice the distance! This conclusion ignores launch errors and wind errors. It simply addresses the hit error caused by mis-estimating range.

Table 11-1 is a compilation of arrow trajectory data computed on the assumption that no air friction existed. These assumptions and data would be correct if astronauts were to shoot arrows on the moon, with two exceptions: First, the strength of gravity, which on earth is 32.2 ft/sec^2 would be much less on the moon. Second, on the moon, an arrow would hit the target aimed upward at the same upward elevation it had at launch because the fletching would be useless, having no air with which to steer.

Although the data is based on zero air friction, the computed "miss" distance for a 10% error in the estimated range will be almost exactly correct. The 180 fps is about what to expect from a 56# compound hunting bow. The 160 fps is about what to expect from a 44# compound hunting bow. Note that at 180 fps when shooting at a target actually 40 yards away but guessed to be either 44 or 36 yards away (due to a 10%-of-range error), the miss distance is 0.72 feet and that at 160 fps it is 0.91 feet. Thus the faster arrow puts hits 0.91 - 0.72 = 0.19 feet = 2.3" closer.

Influence of arrow velocity:

Pins are close together on a fast bow and far apart on a slow bow. Thus, obviously, the miss due to an error in range is less with a fast arrow than with a slow one. The ability of speed to compensate for errors in range is very significant, which is one of two reasons why bowhunters like to shoot the fastest bow they can handle. The other reason, of course, is that a fast arrow penetrates better.

Influence of range:

Elsewhere in this book the author explained why he recommends that a 10-yard pin be set. It is the highest pin ever needed. It is also the pin which is used without correction over the longest span of ranges. Typically, the 10-yard pin is used over a range of from about 8 yards to about 14 yards or so, without correction. Why? It has to do with the geometry of the archer's eye being above the arrow by an amount which is significant at short range but is not significant beyond about 20 yards. The result is that pins start very low (at the arrow's tip, in fact) for zero range and rise rapidly as range is increased out to about 8 yards. Then, from about 8 to 14 yards, the pin stays at a nearly constant height. As the range is increased, the pins start to drop again, arriving finally back at the arrow tip.

Thus, if the actual range is somewhere near 10 yards, a major mistake in estimate of range causes very little error. A 20% error in estimated range would mean that the archer estimated 8 or 12 yards when it was actually 10 yards. No significant miss would result.

If the actual range is 50 yards and the archer miss-estimates by 20%, he would use his 40-yard or 60-yard pin. A total miss would result unless the target were quite large or the arrow extremely fast.

Estimating range:

The use of a range finder is the best way to ascertain range. The author nailed a jack rabbit at 73 yards last year, using a range-finder. The author's average error in estimating range is about 10%. A 10% error at 73 yards would have resulted in a total miss.

A range finder is almost mandatory for practicing range estimation. Make it a habit to walk around the block a few times a week, taking the range finder. At intervals, stop and estimate the range to something. Then take the range with the range finder. Then "memorize" that the actual range of whatever you are looking at is whatever the range finder says. Make a regular game of it, by writing down the "guess" and "actual" ranges. This procedure would make excellent and meaningful competition with other archers.

The local football field is perfect for calibration of a range finder. It is already marked at intervals. Practice in the use of the range finder at the football field gives confidence in the accuracy of the device. The use of a range finder does involve some proficiency. It also requires good eye-sight. The author, whose vision is about 20/40, must be wearing glasses in order to get good accuracy. Also, lighting must be at least fair.

While calibrating the range finder at a football field, stand on the sideline. Note that the sideline looks like two crossing lines, with the point at which they cross being the range at which the finder is set. Doing this gives a better understanding of the precision (or lack of it) of the device. See Figure 11-3.

Another way to stay in practice at range estimation is to leave the range finder accessible in your car. When waiting at stop signs, use it. In order to not be caught napping when the light turns green, take the range first and then guess the range. Be *careful about trying to take a range through the curved glass of a car's wrap-around front window. The results can be wrong, as light is bent differently depending upon which part of the window each eyepiece looks through.

Left-right effects of (known) range:

Left-right accuracy is a lineal function of range. Double the range, double the error. If an arrow is shot exactly at the target, it will hit a vertical line through the target regardless of range and regardless of arrow speed.

Left-right effects of wind:

When an arrow is launched, it immediately lines up with the airstream, thereby automatically "compensating" for crosswind. The path of a friction-free arrow will not be changed by a crosswind. The phrase "friction-free" as used here assumes no fore-and-aft resistance but assumes that fletching has enough side-ways resistance to steer the arrow.

Real arrows have lots of friction. Typical accelerations caused by friction of real arrows are in the neighborhood of 1.0-g's. That is, an insect riding inside a typical arrow will experience a deceleration force of about 1.0 x gravity, with "down" being the arrow tip and "up" seeming to be the arrow nock.

To calculate the cross-wind effect of friction is not easy. In fact, it is so tough that I'm not going to try it. None-the-less, certain maximum influences can be stated, just to put a boundary on the answers. Look at it this way: When the arrow has lined up with the relative wind immediately after being launched, the rear-ward acceleration (as viewed by an observer riding the arrow) can be divided by vector analysis into components. The sideways component of acceleration is what causes wind error. The acceleration drops off until sideways speed equals cross-

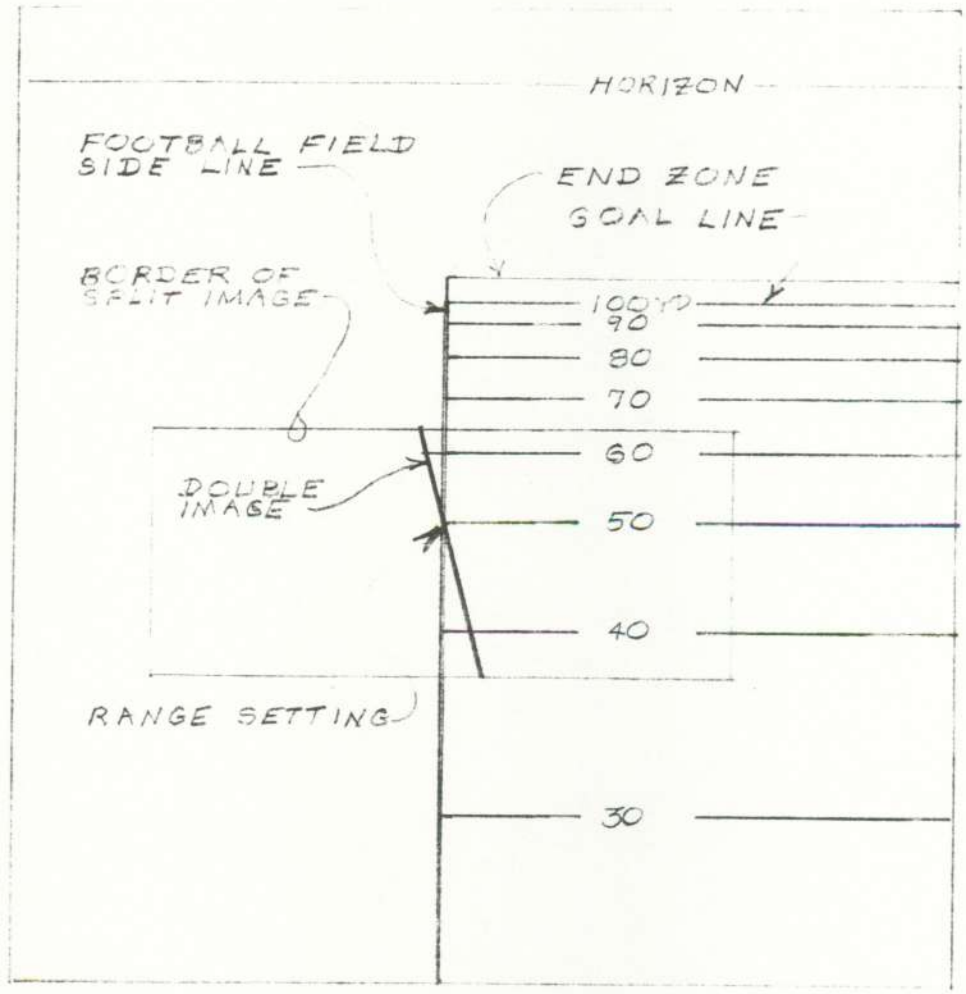


Figure 11-3 ... VIEW OF FOOTBALL FIELD FROM GOAL LINE ALONG SIDELINE AS SEEN THROUGH RANGE FINDER wind speed. Thus the maximum and thus limiting wind error would that the same as though the arrow rode the wind sideways at full cross-wind speed for the entire flight. A 200 fps arrow shot 300 feet has a flight time of about 1.5 seconds. A 20 mph cross wind is a 29.3 fps wind; and in 1.5 seconds it will cover 44.0 feet. Real arrows would have much less error than this, however.

An approximation of wind error can be made by assuming that initial acceleration prevails for the entire flight. Use the formula:

$$\text{Offset} = 1/2 \times a \times t^2$$

Example:

Assumptions:

Arrow - 2117 x 28 with 5" 3-fletch
Weight = 527 grains.
Launch velocity = 200 fps.
Drag at 200 fps = 386 grains.
Cross wind = 20 mph = 29.3 fps.

Calculations:

Initial deceleration = 386/526
= 0.73 g's
$\times\ 32.2\text{ft/sec}^2/\text{g} = 23.6\text{ft/sec}^2$
Sideways fraction = 29.3fps/200fps
= 0.15
Sideways acceleration =
$0.15 \times 23.6\text{ft/sec}^2 = 3.46\ \text{ft/sec}^2$
Duration = 300 ft/200ft/sec
= 1.5 seconds
$\text{Offset} = 0.5 \times 3.46\text{ft/sec}^2\ (1.5\text{sec})^2$
= 3.89 ft.

This answer is probably not too far wrong. The true answer will be <u>less</u> because the assumption of continuous crosswise acceleration is not correct. Cross-wise acceleration will diminish throughout the flight as the arrow slows down.

My "Pro Range Finder 80/2" lists errors as follows:

20 mph:
100 yards = 68".
50 yards = 16".
20 yards = 2".
10 mph:
100 yards = 34".
50 yards = 8".
20 yards = 1".

I know not what assumptions they made, or if their numbers are measured results. The example calculation come up with 3.89 feet, or 47", for 20 mph @ 100 yards. This is in the same ballpark as the 68" listed above, so I can accept their figures.

Concluding comments about cross-wind effects are:

1) Streamlined arrows are less effected than those having heavy drag.
2) Fast arrows are less effected than slow arrows.

<u>Vertical errors due to known range</u>:

"Known" range is a measured range, known to the archer. The archer is assumed to have sighted in his bow for that range. Errors due to known range are lineal with range. Doubling the range will double the error caused by a given mistake in launching. Arrow velocity makes (almost) no difference. The only technical reason that a fast arrow might have less error would be that it's total length of flight is slightly less than that of a slower arrow.

Accuracy as influenced by errors in range estimation

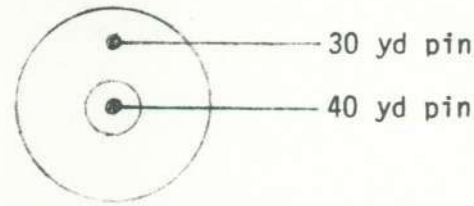
If an archer can keep 9 out of 10 arrows in a 4" circle from 30 yards, what will his group look like when he has to estimate the range? It depends, obviously, upon how accurately he estimates the range. Or at least the up-down grouping does. The left-right accuracy is independent of estimation errors.

GROUP SHOT FROM KNOW DISTANCE

GROUP SHOT FROM ESTIMATED DISTANCE

An archer will never shoot a vertically dispersed group like the one illustrated because, after the first shot, he will know the actual range rather accurately. A vertical group might very well be shot by a succession of different archers each of whom comes within sight of the target, estimates its range, and shoots one shot.

The effect of mis-estimating range can be ascertained very simply by noting where each pin sight points on a target. Say, for instance, that estimated range is 40 yards, actual is 30 yards. The 40 yard pin will be on target; the 30 yard pin will be on impact point.

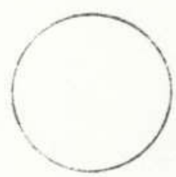


SIGHT PICTURE

The various groups may look as follows:

GOOD SHOT,
GOOD
ESTIMATOR

GOOD SHOT,
POOR
ESTIMATOR



POOR SHOT,
GREAT RANGE
ESTIMATOR

POOR SHOT,
POOR RANGE
ESTIMATOR

A pertinent point to notice is that a poor shooter who is a great range estimator may very well outshoot a good shooter who is a poor range estimator. For methods of calculating exactly what the above groups should look like, see Chapter 16, "Accuracy Statistics".

VELOCITY DETERMINATION

Archery on the moon:

On the moon there is no air and thus no air friction. Were there no air friction on earth, the following formulas would apply exactly. They could be arranged to ascertain initial arrow velocity. "Initial" velocity means what it says, but on the moon initial and final velocities would be the same for a target at the same elevation as the launch point. On the moon, the arrow's speed at the top of it's trajectory will be slower than it's initial velocity. Shooting downhill, speed will increase continuously. On an earth with air, friction will cause final velocity to be slower than initial speed when target and launch point are at same elevation. Shooting downhill with air friction, velocity might increase or decrease, depending upon initial speed, friction, and arrow weight. The moon-applicable formulas are:

$t = 2V/g$ = round-trip time of a vertical shot.

$R_{,max} = V^2/g$ = maximum range of arrow shot at 45°.

$A = \frac{1}{2} \arcsin(RG/V^2)$ = angle above horizontal of shot.

$h = (V^2/2g)\sin^2 A$ = height of zenith.

Vertical shots:

The first formula give the time of flight of an arrow shot straight up. By using a stop watch to measure elapsed time from time of launch to return to surface, velocity can be computed. Solving the formula for velocity yields:

$$V = tg/2$$

Example: Elapsed time = 10.00 seconds.

$$V = 10 \text{ sec} \times 32.2 \text{ ft/sec}^2 / 2$$
$$= 161 \text{ ft/sec.}$$

Note the I've used earth's value of gravity, which is 32.2 ft/sec^2. The moon's value of gravity is smaller because the moon weighs less than the earth.

I've not been able to compute the corresponding formula for arrows shot straight up in air. It is not even clear to me if, for the same initial speed, the elapsed time should be longer or shorter. The time to go up to the top will be shorter in air due to the slowing of the air, which will stop the arrow earlier. But the time to return to earth will be lengthened in air due to the parachute effect.

Maximum range:

The second formula tells how far an arrow will travel on the moon when fired 45° above the horizon. Solving for V yields:

$V = (gR)^{\frac{1}{2}}$ = speed of arrow shot 45° from horizon.

Example: Range = 400 yards.

$$V = \left(400\,\text{yd} \times 3\frac{\text{ft}}{\text{yd}} \times 32.2\frac{\text{ft}}{\text{sec}^2}\right)^{\frac{1}{2}}$$
$$= (38{,}640 \text{ ft}^2/\text{sec}^2)^{\frac{1}{2}}$$
$$V = 197 \text{ ft/sec.}$$

With air friction, an arrow will obviously not fly as far. In "The Technical Side of Archery" it is shown that an elevation of 42° above the horizon yields maximum range in air with real arrows.

The formula does tell you that the arrow's launch speed was faster than the computed value. It does not say how much faster.

Zenith

The last two formulas can be used to compute what the arrow's zenith would be in an air-free environment. By firing first at the full range and then at half the range, the zenith height can be measured.

Example:

Use the 80-yard pin at 80 yards.

Use the 80-yard pin at 40 yards which is half way.

Measure the difference in arrow group heights. This would be the zenith height for an 80-yard shot in an air-free environment.

Construct a table of zenith versus velocity using formulas:

First point: R = 80 yards. V = 200 fps.

$A = \frac{1}{2}\arcsin(RG/V^2)$ = angle above horizontal of shot.

$= \frac{1}{2}\arcsin\left(\frac{80\,yd \times 3\,ft/yd \times 32.2\,ft/sec^2}{(200\,ft/sec)^2}\right)$

$= \frac{1}{2}\arcsin(0.1932) = \frac{1}{2} \times 11.14°$

$= 5.57°$ = launch angle.

$h = (V^2/2g)\sin^2 A$ = height of zenith.

$= \frac{(200\,ft/sec)^2}{2 \times 32.2\,ft/sec^2}\sin^2 5.57°$

$= 40{,}000\ ft/64.4 \times (0.9706)^2$

$= 621\ ft \times 0.00942$

$= 5.85$ ft = zenith height of 80-yard shot.

Additional points can be calculated, and the velocity corresponding from any given zenith height can be interpolated. See table. Table lists, also, the values for 40 yards and 20 yards.

In the real world of archery with air friction, the zenith will have to be just a little higher in order for an arrow of 200 fps to reach 80 yards. Thus the actual velocity is slower than the computed.

When the three methods of computing velocity without friction given so far are used uncorrected in air, one (the vertical shot) gives indeterminate results, another (the maximum range shot) proves that the arrow was launched at least as fast as computed, and the third (the zenith-height measurement) also proves that the arrow was launched at least as fast as computed.

Pin Gap

As all archers who use pin sights know, pins of fast arrows are close together and pins of slow arrows are far apart. The space between pins defines an angle. If a peep sight embedded in the bow string is used, the angle is from one pin to the peep sight and back to the next pin. Otherwise, the angle is from one pin to the eye and back to the next pin. The angle from the horizontal which an arrow must be fired in a frictionless environment is given by:

$$A = \tfrac{1}{2}\arcsin(Rg/V^2)$$

By calculating that angle for each pin and then taking the difference between angles, the angle between pins can be calculated. The sine of the angle between pins multiplied by the distance from the peep sight to the plane of the pins is the distance between pins.

Example: Arrow speed is 200 fps.
High pin is for 40 yards.
Next pin is for 60 yards.
Peep-to-pins is 28.75".

$$A_{,60} = \frac{1}{2}\arcsin\left[\frac{60 \times 3\,ft \times 32.2\,ft/sec^2}{(200\,ft/sec)^2}\right]$$

$$A_{,60} = \tfrac{1}{2}\arcsin(180 \times 32.2/40{,}000)$$
$$= \tfrac{1}{2}\arcsin 0.1449 = \tfrac{1}{2} \times 8.3315^\circ$$
$$= 4.166^\circ$$

$$A_{,40} = \frac{1}{2}\arcsin\left[\frac{40 \times 3\,ft \times 32.2\,\frac{ft}{sec^2}}{(200\,ft/sec)^2}\right]$$

$$= \tfrac{1}{2}\arcsin(120 \times 32.2/40{,}000)$$
$$= \tfrac{1}{2}\arcsin 0.0966 = \tfrac{1}{2} \times 5.5434^\circ$$
$$= 2.772^\circ$$

$$A_{,60} - A_{,40} = 4.186^\circ - 2.772^\circ$$
$$= 1.394^\circ$$

sine of 1.394° = 0.02433

Distance between pins = 28.75" x 0.02433 = 0.70".

By repeating such calculations, arrow velocity for any given pin spread can be tabulated. See table for the tabulated spread between 30-yard and 40-yard pins.

This technique is correct in a friction-free environment only when the parallax caused by the eye's being above the arrow is negligible. The parallax is very significant out to 20 yards and somewhat significant between 20 and 30 yards. Thus this procedure should not be used unless parallax is also computed under 30 yards.

TABLE OF VELOCITY PARAMETERS

V (fps)	R,max (yds)	40/20 (in)	80/40 (ft)	Time (sec)	30/40 gap (inches)
100	104	72.4	-	6.21	1.498
110	125	59.0	21.67	6.83	1.196
120	149	49.2	17.46	7.45	.982
130	175	41.7	14.52	8.07	.823
140	203	35.8	12.33	8.70	.700
150	233	31.1	10.63	9.32	.603
160	265	27.3	9.27	9.94	.525
170	299	24.2	8.17	10.56	.461
180	335	21.5	7.26	11.18	.408
190	374	19.3	6.50	11.80	.363
200	414	17.4	5.85	12.42	.324
210	457	15.8	5.30	13.04	.292
220	501	14.4	4.82	13.66	.263
230	548	13.2	4.41	14.29	.239
240	596	12.1	4.04	14.91	.217
250	647	11.1	3.72	15.53	.198
260	700	10.3	3.44	16.15	.181

To ascertain arrow velocity:

1. Shoot an arrow as far as possible. Measure range. Enter "R,max" column; read velocity in left column. Actual launch speed is faster.
2. Shoot straight up (75° to 90°). Clock arrow's time of flight. Find time in "Time" column. Read velocity in left column. Actual launch speed is indeterminate.
3. Sight in bow at 40 yards. Shoot at 20 yards. Measure how high above target arrows hit. Enter "40/20" column. Read velocity in left column. Actual launch speed is faster.
4. If velocity per "40/20" method is greater than 200 fps, use "80/40" column by sighting in at 80 yards and then shooting at 40 yards. Actual launch speed is faster.
5. Measure distance between your 30 yard pin and your 40 yard pin. Find that difference in "30/40 gap" column. Read velocity in left column.

Notes:

All numbers ignore air friction.
All except "30/40 gap" column are true for any bow, any arrow.

"30/40 gap" column assumes distance from peep sight to plane of sighting pins is 28.75". To compensate for your bow, multiply your gap by length applicable to your bow and divide by 28.75". If you do not use a peep, use distance from eye to plane-of-pins.

Chapter 13 ... Penetration

The penetration of an arrow into the body of an animal is a topic of much interest to bowhunters. First, let's make some obvious statements, such as:

- o A sharp arrow penetrates better than the same arrow when dull.
- o A heavy arrow penetrates deeper than a light one of the same diameter if both arrive at the same velocity.
- o An arrow traveling fast penetrates deeper than the same arrow traveling more slowly.

A somewhat less obvious statement is:

- o A straight-flying arrow penetrates better than one not flying straight.

Archer's Decisions

Of the four parameters mentioned (sharpness, weight, speed, straight flight), the archer has some obvious control, such as:

- o He can sharpen his broadheads.
- o He can get better speed and/or weight by shooting his bow at heavier poundage.
- o He can get straighter flight by tuning his bow.

Assuming he has done the above, the question remains as to the trade-offs of different arrows shot from the same bow. In particular, the "overdraw" conversion decision. An "overdraw" is an arrow rest located 3 or 4 inches aft of the conventional arrow rest. With the arrow rest aft, a shorter arrow can be shot. In Chapter 9 the ability of an arrow to tolerate launching forces is shown to be inversely proportional to the square of the arrow's length. Thus a 26" arrow can tolerate a ratio of forces equal to (29/26) squared or 1.244 or 24.4% more. But since the same bow is being used with or without "overdraw", the arrow has no need to tolerate more force. Thus the arrow's wall thickness may be reduced. The shorter arrow is then lighter in weight not only because it is shorter, but mainly because it's wall can be thinner. The before-and-after data for the author's Martin Tiger bow was:

Table 13-1.

Comparative Data for Overdraw Conversion

Bow is a Martin Tiger compound of peak draw of 69 pounds.

	Conventional	Over-draw
Arrow name	2117	2114
Diameter	21/64"	21/64"
Wall thickness	0.017"	0.014"
Length	29"	26"
Weight = m/g =	545 grains	460 grains
Velocity = V =	193 ft/sec	205 ft/sec
Momentum = mV =	15.0 ft-lb/sec	13.5 ft-lb/sec
Energy = $\frac{1}{2}mV^2$ =	45.0 ft-lb	42.9 ft-lb

Practical conclusion: **A heavier arrow will penetrate deeper than a light arrow when shot from the same bow.** This will remain true regardless of whether penetration is proportional to momentum or to energy.

Thus the hunter is right back where he started: speed is helpful in offsetting range estimation errors; weight is helpful for penetration and smooth shooting. The tradeoff depends upon the game sought. For deer, I will stay with the high speed of the overdraw. For anything bigger, I'll use the heavier arrow and simply shoot from a shorter range. How much shorter? See the chapter on accuracy.

Momentum versus Energy:

One should first get the answer to the question, "Which is penetration proportional to; arrow energy or arrow momentum?" The literature is divided on the question. Burton G. Schuster in "Ballistics of the Modern Working Recurve Bow and Arrow", American Journal of Physics, April 1969 says that penetration is proportional to momentum. His assumption, however, is that "the force on the arrow head is proportional to the velocity, and that the drag on the shaft is proportional... to the depth that the shaft has penetrated." He goes on to say, based on those assumptions, that "For a given bow, the penetration is then proportional to $(2MV)^{\frac{1}{2}}$." In the chapter on air drag, it was shown that air drag depended upon velocity raised to the 1.5th to the 2.0th power. How Schuster concluded that flesh drag depended upon velocity raised to the 1/2 power is unknown. Maybe he is right; maybe not.

Every scenario presented later in this chapter leads to the conclusion that penetration is a function of foot-pounds of energy, not of momentum.

Categories of Resistance and Resulting Decelerations:

An arrow is slowed by tip resistance and by wall drag. The categories of tip resistance and wall drag vary. The categories are:

Tip Resistances:

a) Cutting work.
b) Sliding friction.
c) Impact.
d) Displacement work.
e) Increases of above caused by untrue flight.

Wall Drag:

a) Constant drag associated with unchanging wall contact area, such as when shooting through a thin target.
b) Increasing drag when drag area increases with depth of penetration, such as when shooting into a bale of hay.
c) Increases of above caused by untrue flight.

Analysis:

What I'd like to do in the following paragraphs is to analyze each of the above forms of resistance separately. The reader should keep in mind that a real arrow responds to all of them at the same time. How close the assumed scenario applies to the archer's arrow depends upon how close the assumptions fit the facts.

Tip Cutting Resistance:

What I'm about to describe is my own theory, based mainly upon what seems logical to me.

When a broadhead presses against a material, the material "gives" in a stretching manner until it breaks. The amount of material involved depends upon sharpness. A sharp blade stretches only the nearest part of the material. A dull blade touches more material and therefore causes more material to be stretched.

The energy consumed in the cutting process depends upon the amount of stretching done before rupture occurs. None of the energy is recoverable. When the material ruptures, it is comparable to a dry fire caused by a broken bow string. Put another way, there is absolutely no tendency for the arrow to bounce. Energy has been used to stretch the material, and all that energy is lost when the material ruptures.

Tip Sliding Friction:

The friction on the sides of a broadhead tip will be computed in the same manner as will the friction of the walls of the arrow shaft, described later in this chapter.

Tip Impact:

The energy imparted by the tip of an arrow to whatever it is hitting and penetrating can vary all over the map. If you shoot an arrow into a pile of rocks, some rocks will fly off in various directions, carrying off kinetic energy. The result as far as the arrow is concerned can vary a great deal. The arrow may do anything between bouncing back to continuing forward. So many variables influence the energy exchange that trying to nail them all down is futile. A few interesting scenarios can be mentioned, however.

a) The Bounce

An arrow with blunt tip shot at a heavy steel plate could bounce. (It's more apt to shatter, but that's a different scenario.) How far it will bounce varies, but the theoretical maximum is one where bounce speed is almost equal to impact speed. Under this scenario, penetration is less than zero and energy is not dissipated.

b) The Merge

If you shoot a small bird and the arrow stays the the bird, you have a "merge" situation. The law of conservation of momentum can be used to compute the resulting speed of the arrow/bird combination. The law says that the arrow's initial momentum will equal the momentum of the arrow/bird combination. Mathematically, the conservation of momentum law says that the weight of the arrow multiplied by it's initial velocity will equal the weight of the combination of arrow and bird multiplied by their common resulting velocity.

$W_A V_A = (W_A + W_B) \times V_{A+B}$ where

W = weight. V = velocity. A = arrow. B = bird.

Assume the bird weighs 1,200 grains and the arrow weighs 500 grains and is going 200 ft/sec. What will the resulting speed be after impact? How much kinetic energy will the arrow/bird combination have?

$600 \text{ gr} \times 200 \text{ fps} = (600 \text{ gr} + 1200 \text{ gr}) \times V_{A+B}$

$V_{A+B} = (600\text{gr}/1800\text{gr}) \times 200 \text{ fps} = 67 \text{ fps}.$

$E_{initial} = W_A V_A^2.$

$E_{final} = (W_A + W_B) \times V_{A+B}^2$

$E_{final}/E_{initial} = (1800\text{gr}/600\text{gr}) \times 67^2/200^2 = 3 \times (1/3)^2 = 1/3.$

This shows that the kinetic energy remaining is 1/3 of the initial kinetic energy. 1/9th is in the arrow; 2/9ths is in the bird, and 6/9ths has been dissipated into the form of heat.

Displacement Work Done by Tip:

Were an arrow to be shot at a spring which was free to compress but not to spring outward again, the spring would be compressed by the incoming arrow. No (major) energy would be lost. No significant friction would be involved. Yet the arrow's "penetration" would be a definite number. When shooting arrows into "springy" material, such as Ethafoam, some of the arrow's energy is transferred to the material in the form of stored mechanical energy. Pulling the arrow out of the Ethafoam is comparable to letting the spring decompress.

The amount of energy stored in the spring which is hit straight on is exactly comparable to the amount of energy stored in a longbow. In fact, were the spring sized correctly, the "curve" of the spring being compressed by the incoming arrow would be an exact mirror of the bow's force-draw curve, assuming both the bow and the spring were 100% efficient.

The amount of energy stored in the Ethafoam is not the same, because the "springs" load at right angles to the arrow's flight. Thus the mechanical energy stored in the Ethafoam is proportional to the depth of penetration. In this paragraph I'm ignoring the frictional energy dissipated in the Ethafoam in the form of heat. The amount of energy stored is also very dependent upon the arrow's diameter. The amount of energy stored in a spring varies with the square of the amount of compression.

Tip Resistance due to Untrue Flight

A number of bowhunters whose opinions I respect assure me that a straight-flying broadhead penetrates better than one which is fishtailing or porpoising. A friend and very experienced bowhunter, Rick Berg of Bowhunters Unlimited of San Jose, guarantees me that his arrows penetrate best when the range is about twenty yards. At that distance, his arrows have steadied up but not slowed significantly. Dan Quillian of Archery Traditions, when told this opinion, agreed 100%. Dave Holt, author of "Balanced Bowhunting" says the same thing.

Why should that be? I presume it is rather obvious, when you give it some thought, that the tip of a broadhead which has a sideways component to it shouldn't cut as well as one going straight. It would be like cutting a piece of meat while holding the knife blade a little sideways.

Constant Drag on Walls

Once an arrow has hit something, it's deceleration is tremendous. Make the assumption than an arrow traveling 200 fps stops in a distance of two feet. Assume, further, that the decelerating force is constant, as when shooting through a thin target. See Figure 13-1. Then ask the question: How many g's does that take? The basic formula is:

$2aL = V^2$, where
: a = deceleration
: L = length to stop (2 feet)
: V = velocity, original (200 feet/second)

Solving for a yields:

$$a = V^2/2L = (200 \text{ ft/sec})^2/(2\text{x}2 \text{ ft}) = 10{,}000 \text{ ft/sec}^2$$

Dividing a by the force of gravity, which is 32.2 ft/sec^2 yields the number of "g" forces:

g's = a/g = 10,000/32.2 = 311 g's. That's a <u>big</u> number!

The actual force working to slow the arrow depends upon the arrow's weight and is given by the formula: F = ma (force = mass times acceleration). Let's assume an arrow weighs 550 grains.

$$F = (550\text{gr}/7000\text{gr}/\#) \times 10{,}000 \text{ ft/sec}^2 / 32.2 \text{ ft/sec}^2 = 24.4\#.$$

That seems like a reasonable number. The force acts for about the same distance as does the bow string while firing. Unlike the bow string, the force is constant until the arrow stops. Thus the 22.2 pound constant force is about the same as a 44.4 pound peak force in a long bow, because the long bow goes from peak to zero over a 24" stroke, averaging 22.2 pounds. Depending upon the bow's virtual mass, the bow's draw weight would be in the range of 60#.

Interestingly, I tried pulling a few arrows out of my ethafoam target, measuring the pull on a 100# spring scale. The break-away forces were in the neighborhood of 25 pounds, but the steady pull (sliding) forces happened too quickly to get a measurement. The coefficients of static friction are higher than those of sliding friction.

The next question is, how does the 24.4 pounds of resistance get applied to the arrow? The real answer is quite complicated unless you make some simplifying assumptions. Let's assume that the arrow has penetrated a 2" thick broadhead ethafoam target. The friction thereafter is entirely wall friction. The tip is in free air and not, therefore, meeting any resistance. The basic formula for sliding friction is:

F = KAp = coefficient of sliding friction x area x pressure.

To ascertain the coefficient of sliding friction between the ethafoam and the arrow wall could be done rather easily. I've not done it, however, so I'll have to guess. My guess is that the coefficient is about 1.0. We know the force. We know the area. The pressure exerted by the ethafoam can, therefore, be computed, as follows:

$A = \text{pi} \times D \times 2" = 3.1416 \times (21/64)" \times 2" = 2.06$ sq.in.
$F = 24.4$ pounds.
$p = F/KA = 24.4\#/(1.0 \times 2.06\text{in}^2) = 11.8\ \#/\text{in}^2$.

This 11.8 psi is a reasonable number.

This constant deceleration analysis is probably close to correct for an arrow penetrating an ethafoam target, where most of the resistance is wall drag over a small length. The same analysis would apply, perhaps, to an arrow penetrating into an animal's body cavity via thick hide. The soft membranes within the body cavity provide very little resistance, whereas the thick hide keeps exerting drag over a small length of the arrow's wall. Incidentally, this might be an excellent argument for using a three-or-more bladed broadhead rather than a two-blade broadhead. It would seem to me that a hide penetrated by a 3-blade head would not be able to apply strong wall pressure, whereas one opened by a 2-blade would.

The depth of penetration where deceleration is caused by a constant wall force is directly proportional to an arrow's energy, not it's momentum. This can be proven by noting that work is equal to force times distance. (Work and energy are the same thing.) Solving for distance (penetration), we get:

$d = E/F$, where d = depth of penetration (distance)
E = energy of arrow
F = force (constant)

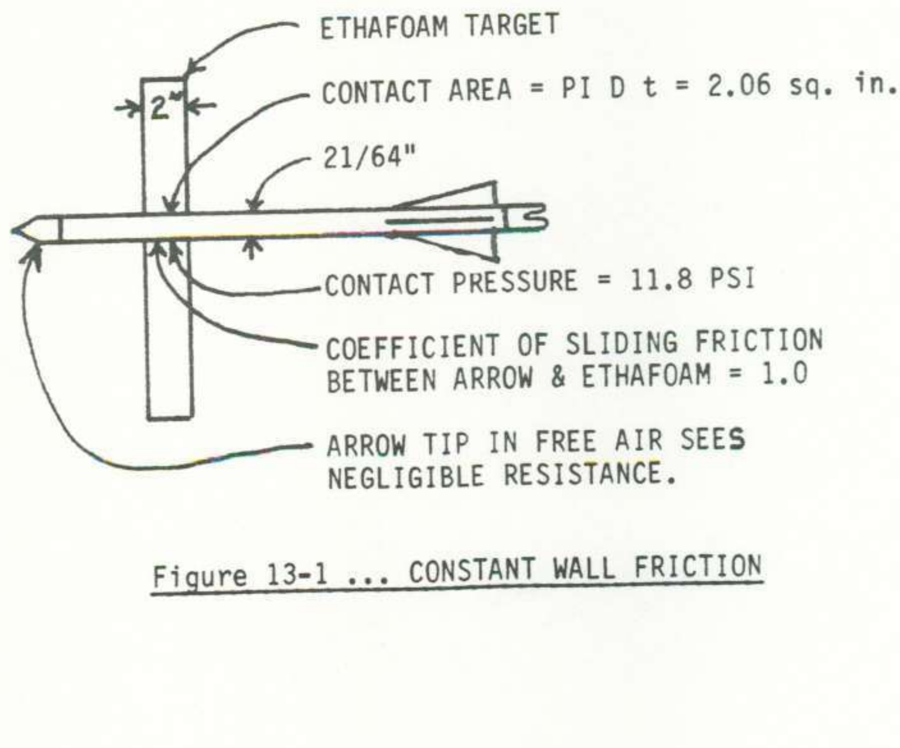


Figure 13-1 ... CONSTANT WALL FRICTION

Increasing Drag on Walls with Depth of Penetration

An arrow shot into a bale of hay presumable encounters increasing resistance as it goes deeper. If the resistance met by the head could be ignored, and if the wall resistance were equal for every square inch of wall area, then the resistance would increase linearly with each inch of arrow penetration. The forces involved would be very much the opposite of the forces applied by a longbow during launch. In fact, were the dept of penetration equal to the stroke of the bow, the slowing forces would be the exact mirror of the launching forces.

The depth of penetration would, again, be proportional to the arrow's energy, not it's momentum.

Wall Drag Change Due to Untrue Flight

The circumstance under which I have noticed that an arrow not flying straight fails to penetrate as far is when tuning my bow with unfletched arrows. I've even bent arrows by shooting them into a bale of straw without fletching. It should be intuitively obvious that an arrow having a sideways component of velocity will experience greater resistance than one flying straight.

Statements:

In each of the scenarios mathematically analyzed in this chapter, penetration was shown to be proportional to arrow energy. In no scenario was it proportional to momentum. Thus I will side with those authors who say penetration depends upon energy. That being the case, the following statements can be made:

"Penetration is directly proportional to arrow weight."

"Penetration is proportional to velocity squared."

"Where wall drag is predominant, penetration is inversely proportional to arrow diameter."

Bullets vs Arrows

Dan Quillian of Archery Traditions reports that arrows penetrate sand bags and/or water deeper than do bullets. He's talking about steel-jacketed bullets weighing 150 grains traveling 2,800 ft/sec. And he is talking about arrows having tips shaped identically to the bullet tips, weighing 600 grains and traveling 173 ft/sec. Let's calculate the comparative energy and momentum. Recall that energy equals 1/2 x mass x

$E = \frac{1}{2}W/gV^2 = (150gr/7000gr/\#)(2{,}800ft/sec)^2/(2 \times 32.2ft/sec^2)$
E = 2,608 ft-lb. for a bullet weighing 150 gr traveling 2800 fps.

$E = \frac{1}{2}W/gV^2 = (600gr/7000gr/\#)(173ft/sec)^2/(2 \times 32.2ft/sec^2)$
E = 40 ft-lb for an arrow weighing 600 gr traveling 173 fps.

$M = (W/g) \times V = (150gr/7000gr/\#/32.2ft/sec^2) \times 2{,}800\ ft/sec$
M = 1.86 lb-sec for a bullet weighing 150 gr traveling 2800 fps.

$M = (W/g) \times V = (600gr/7000gr/\#/32.2ft/sec^2) \times 173\ ft/sec$
M = 0.46 lb-sec for an arrow weighing 600 gr traveling 173 fps.

Recapitulation

	Energy (ft-lb)	Momentum (lb-sec)	Weight (grains)	Speed (ft/sec)
Arrow	40	0.46	600	173
Bullet	2,608	1.86	150	2,800

Obviously, since the bullet carries much more of both energy and momentum and has the same shape tip, one would expect the bullet to penetrate more deeply. I don't pretend to fully understand what the differences are, but my seat-of-the pants feeling is that the form of energy transfer is much different between bullet and arrow. The bullet, while traveling fast, is delivering it's energy to the sand bag by impact mechanisms. By the time the bullet slows to the arrow's velocity, two things give it less penetration power. One is that it may not be traveling straight. The second is that weighs much less than the arrow.

*** END ***

ARCHERY SIGHTS

The modern bow uses sights at the front of the bow. Typically, five pins are set, each for a different range. Many bows are also equipped with a "peep" sight, which is an orifice imbedded within the bow's string through which the archer looks. The peep sight is needed by those archers who have difficulty anchoring in exactly the same position every time. See Figure 1.

Highest pin:

The geometry of archery is such that there is one pin location which is higher than any other, regardless of range. Usually the range for which this highest pin is correct is between 7 and 10 yards. Ranges shorter than or longer than 7 to 10 yards use lower pins. See Figure 2.

Recommended yardages for pin settings for hunting &/or field shooting:

Always set the highest pin at 10 yards. Why? First, it is an easy number to remember. Second, it is the only pin which is correct over a wide range, extending from about 6 yards to about 13 yards. See Figure 2. Although theory would say to use (perhaps) 8 yards, the 10 yard pin is almost exactly the same as the 8 yard pin. By setting the 10 yard pin, which is just a tiny bit lower than the 8 yard pin, the range over which the 10 yard pin is used is extended. Typically, the 10 yard pin is the correct pin to use when the target is anywhere between 6 yards and 13 yards.

Set the second pin at 20 yards. Why? When shooting at targets which are at ranges between pins, the archer must "gap". For instance, if the target is 60 yards off and the pins are set at 50 and 70 yards, the archer splits the distance between the pins. Beyond a range of about 30 yards, the geometry of archery is such that linear interpolation between pins works correctly. But it does not work correctly between 10 yards and 20 yards. Why? The geometry of the bow and arrow puts the arrow below the eye and sights by an amount which is significant at short ranges (under 20 yards) but which is not significant at longer ranges (over 20 yards). Thus setting a pin at 20 yards does two things. First, it provides an upper end of ranges which can be "gapped" in a linear fashion. In other words, to hit a target at 25 yards, put the target half way between the 20 yard and 30 yard pins. Second, the 20-yard pin delineates the lower end of ranges which must be gapped in a non-linear way. Half way between the 10-yard and 20-yard pins is not correct for 15 yards; rather it is correct for about 17 or 18 yards.

Put the next three pins either at 30, 50 & 70 yards or at 40, 60 & 80 yards. The decision should be based upon your own preferences. The author, for instance, has a 30-yard range in his backyard, so he uses 30, 50, 70. Do not use more than 6 pins. They are not necessary and tend to confuse the shooter with too many pins.

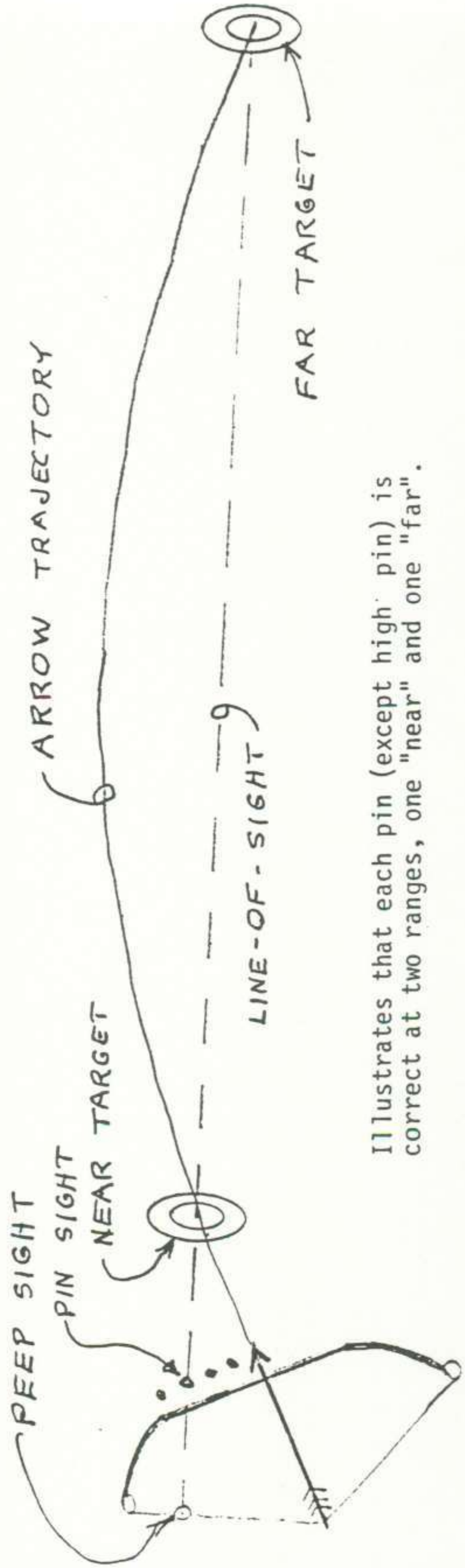
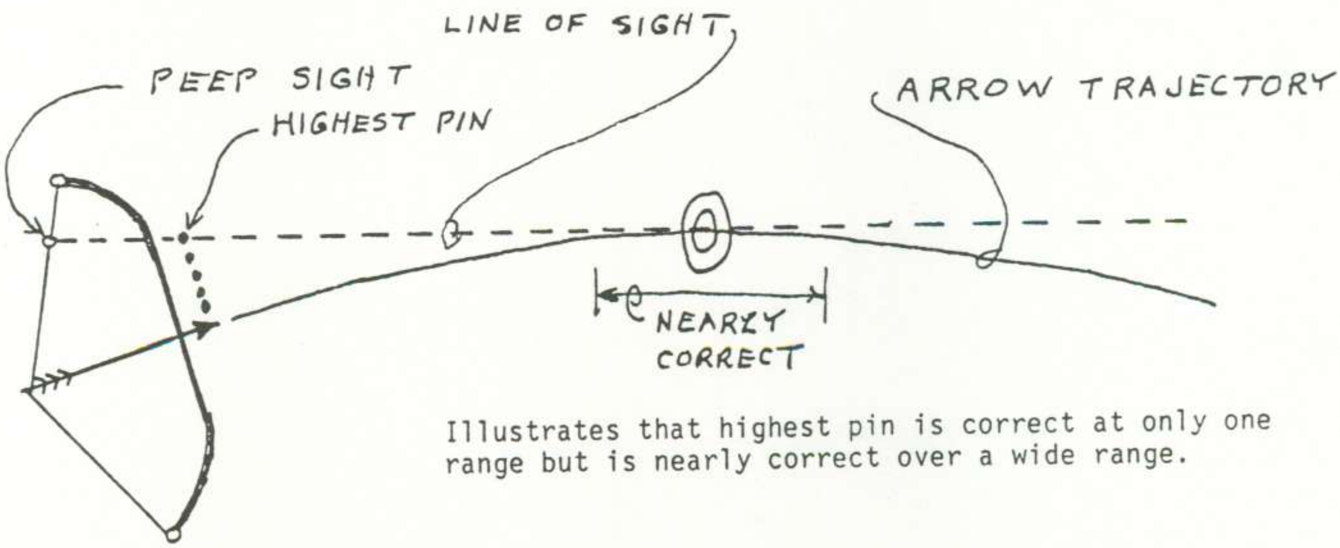


Figure 1 ... Bow Sight Geometry: two ranges per pin



Illustrates that highest pin is correct at only one range but is nearly correct over a wide range.

Figure 2 ... Bow Sight Geometry: Highest Pin

The author does not heed his own advice exactly. He has two compound bows, one a 55-pound Martin Cougar target bow and the other a 69-pound Martin Tiger hunting bow. The 55-pound bow has pins at 10, 20, 30, 50 & 70 yards (per above recommendations). The 69-pound bow has pins at 10, 30, 50, 70 & 90 yards. Why no 20-yard pin? Because the author's single-plane sight will not permit pins set as close together as needed for 10-yards and 20-yards.

Very close ranges:

If you want to shoot a rattlesnake at 2, 3 or 4 yards, what pin do you use? You ought to know. The 20, 30 and longer pins are also correct at much shorter ranges. See Figure 1. The author's target bow's 20-yard pin is also correct at 11 feet, and it's 30-yard pin is correct at 6 feet. The author's hunting bow's 30-yard pin is correct at 8 feet. As a practical matter, the author recommends that you learn by trial-&-error for what (very close) ranges your 20 and 30-yard pins are correct. If the range is closer than 6 to 8 feet, don't shoot the arrow; stab with it!

See Figure 1.

Calculation of Sight Pin Locations (for horizontal shots)
See Figure 3.

Assumptions:

1. Air friction is non-existent.
2. Pin sights are in a plane perpendicular to arrow.
3. A peep sight is imbeded within bow string. If no peep sight is used, the eye's location should be used instead.

Definitions:

1. Range is the horizontal distance to the target from the tip of the arrow.
2. Pin sight reference point is as high above the arrow as is the peep sight.
3. Parallax is the angle between the peep, the target, and the arrow.

Pin Sight Settings:

Pin sight locations are computed by considering the combined effects of parallax and of arrow elevation angle.

Parallax: See Figure 4.

Parallax is the angle created by virtue of the archer's line-of-sight being above the arrow. Parallax is maximum at zero range, at which range the archer would look (almost) at the tip of the arrow. "Almost" refers to the fact that the tip of the arrow is not usually in the plane of the range pins (although it could be). The angle at zero range is the angle from peep sight to intersection of arrow with plane of sighting pins and then back along arrow shaft. Parallax angle gets smaller as range increases, going to zero at infinite range. The parallax angle is negative except when arrow is at such high elevation that tip is higher than eye or peep sight.

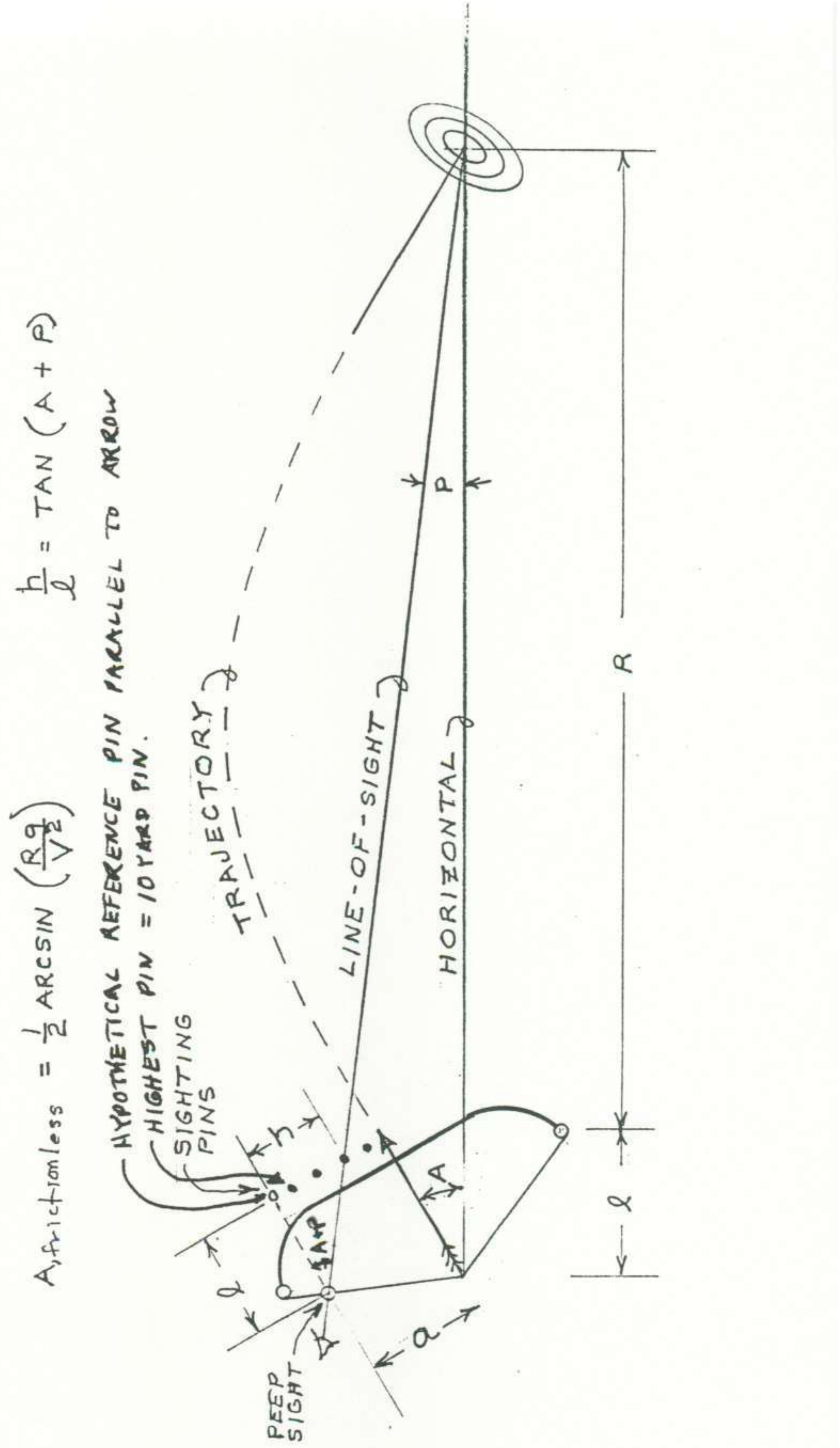


Figure 3 ... Pin Sight Geometry for Horizontal Shots

To visualize the effect of range on parallax, visualize an arrow incorporating a flashlight, the beam of which is finely focused in the direction the arrow points. Point the beam (and thus the arrow) at the target, starting at zero range. Note the pin sight setting. Then back away from the target. Note that the pin sight setting will rise, rapidly at first. At long range, the pin sight will be as high above the flashlight-arrow as is the peep sight. See Figure 4.

Arrow elevation angle:

Ignoring air friction, the formulas describing arrow elevation angle above horizontal when shooting at a target at the same elevation are:

$A = (1/2)\ \arcsin(RG/V^2)$, Equation #6-11

where: R = range

V = initial velocity

A = elevation above horizontal

g = gravity = 32.2 ft/sec^2

Combined Effects: See Figure 3.

The parallax angle diminishes rapidly as the range increases from zero. Typically, it has diminished from 8° at zero range to less than 1° at 10 yards. During this same change of range, the arrow's elevation angle typically changes hardly at all (from zero degrees to somewhat less than one degree). The result is that the proper location for a sight pin when going from zero range outward first rises and then falls. The range which results in the highest sight pin is, as a practical matter, the closest range for which sight pins will be used. On the author's bow, this range is at 8 yards. There is no perceptible change at 7, 8, & 9 yards.

When looking through the peep sight at the pin sight reference point, the line of sight will be parallel to the arrow. The included angle between (imaginary) reference pin and actual pin is the sum of the absolute values of parallax angle plus elevation angle.

Sight locations: The final formula for pin sight locations is:

$h = l \tan[0.5 \arcsin(Rg/V^2) + \arctan(a/(R+l))]$, Eqn. #14-1, where

h = Distance from high reference point in plane of pin-sight. High reference point is as high above arrow as is eye or peep sight.

a = Distance from eye or peep sight to arrow.

l = Distance from eye or peep sight to plane of peep sight.

R = horizontal range from plane of pin-sight to target.

g = gravity = 32.2 ft/sec^2.

V = velocity of arrow as it leaves bow.

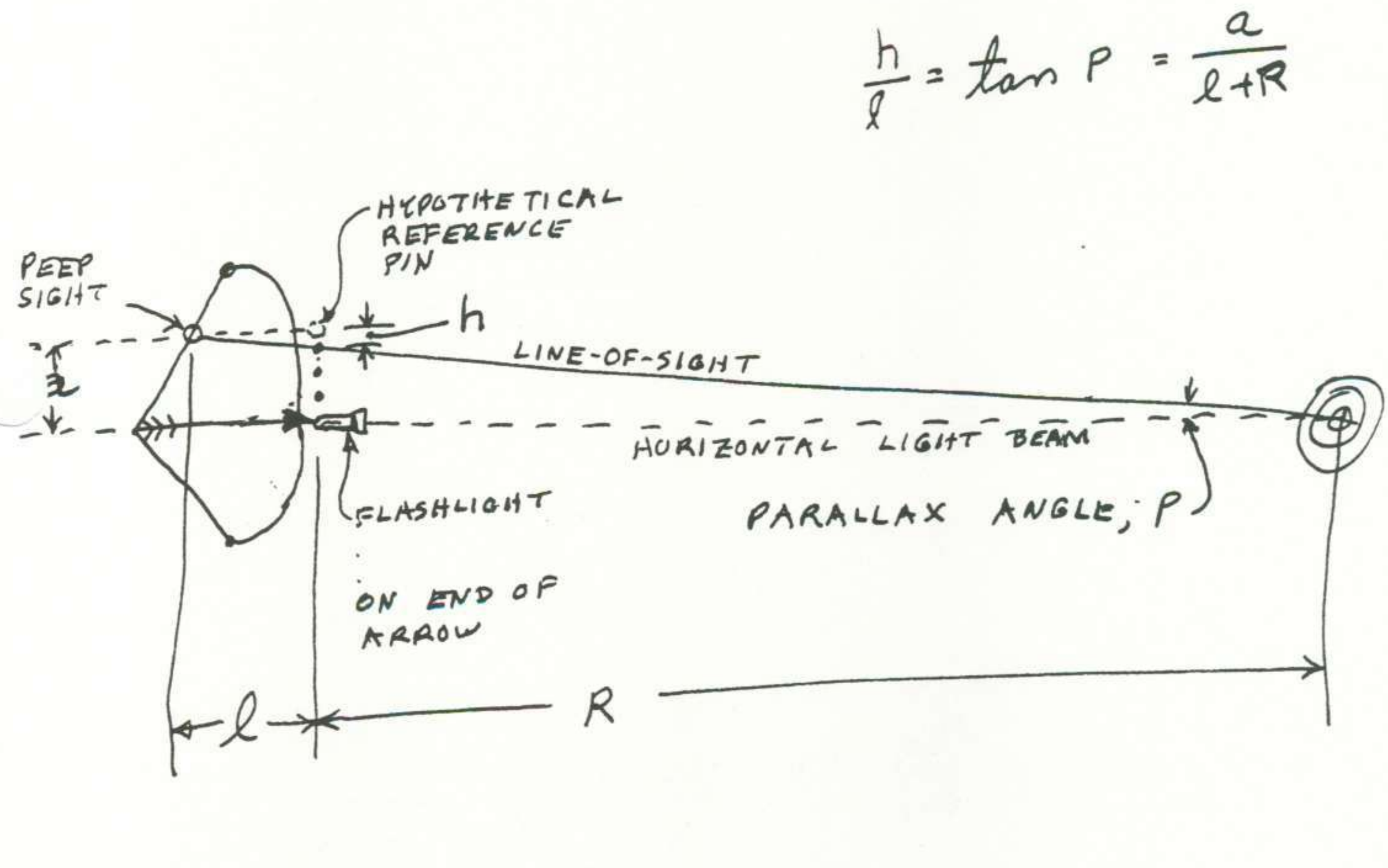


Figure 4 ... Parallax with Flashlight on Arrow

Range at which pin is highest

The formula for the range at which the pin is highest is:

$R = V(2a/g)^{\frac{1}{2}} - 1$, Equation #14-2

This formula is derived by noting that the range at which the pin is highest will prevail when the parallax angle and the arrow elevation angle are equal.

Refer again to Figure 3 for what each parameter in the above formula is.

Inspecting the formula, one can deduce that:

1. Range increases directly with arrow velocity.
2. Range increases as the square root of "a", which is the height of the peep sight above the arrow.

Calculations using the dimensions of the author's bows are:

$1 = 27"$ $a = 3.9"$

$$R = 150\frac{ft}{sec}\left(\frac{2\times 3.9"}{32.2\ ft/sec^2 \times 12\frac{"}{ft}}\right)^{\frac{1}{2}} - \frac{27"}{12"/ft}$$

$$= 150\frac{ft}{sec}\left(0.202\ sec^2\right)^{\frac{1}{2}} - 2.25ft$$

$$= 150\frac{ft}{sec} \times 0.142 sec - 2.25ft$$

$$= 21.3 ft - 2.25 ft$$

$$= 19.1 ft$$

$$= 6.4 yd \text{ (for 150 fps).}$$

Substituting other velocities yields:

Velocity	Range
150 fps	6.4 yards
175 fps	7.5 yards
200 fps	8.7 yards
250 fps	11.1 yards

Substituting other "heights of eye" at 200 fps yields:

Height	Range
0"	-0.8 yards.
1"	4.0 yards.
2"	6.0 yards.
3"	7.6 yards.
3.9"	8.7 yards.
5"	10.0 yards.
6"	11.0 yards.

Distance Between Pins

The formula for pin locations (Equation #14-1) cannot be used in the real world of archery to actually locate a particular pin. The complex dynamics of arrow launch are not recognized by the formula. Spacing between pins can, however, be computed. The correct way to do so for frictionless flight is to calculate the predicted values of "h" at the two ranges of interest and then subtract one from the other, using Equation #14-1. A computer print-out for the author's bow follows:

TABLE 14-1 ... PIN LOCATIONS VERSUS RANGE (Frictionless Flight)

Bow = Martin Cougar   Draw weight = 55#

l = Peep-to-pin plane: 28.75"

a = peep-to-arrow height: 3.88"

V = velocity: 180 ft/sec

Maximum range: 335 yards

Formulas: $R,max = V^2/g$

P = parallax = $a/(R + l)$

A = arrow elevation = $0.5 \arcsin (RG/V^2)$

h = (-) pin height = $l \tan (A + P)$

Range (yards)	Parallax (degrees)	Arrow Elevation (degrees)	(A + P) (degrees)	h (inches)	Change of "h" over previous 10 yards (inches)
0	7.72	.00	7.72	3.90	
2	2.20	.17	2.37	1.19	
4	1.29	.34	1.63	.82	
6	.91	.51	1.42	.71	
8	.70	.68	1.38	.69	
10	.57	.85	1.43	.72	
20	.30	1.71	2.01	1.01	.292
30	.20	2.57	2.77	1.39	.382
40	.15	3.42	3.58	1.80	.408
50	.12	4.29	4.41	2.22	.420
60	.10	5.15	5.25	2.64	.428
70	.09	6.02	6.11	3.08	.434
80	.08	6.90	6.98	3.52	.440
90	.07	7.78	7.85	3.96	.446
335	.02	45.00	45.02	28.77	

Inspecting the "change of "h" over previous 10 yards" shows what might be expected, namely:

1. The change is small (0.29") between 10 and 20 yards.
2. The change is much larger (.38") between 20 and 30 yards.
3. The change reaches a steady number beyond 30 yards, only gradually increasing beyond from 0.41" to 0.45".

It is for the above reasons that the author recommends setting a 10-yard pin (which is the highest pin), a 20-yard pin (which separates the ranges at which linear interpolation between pins is correct) and then either 30/50/70 or 40/60/80.

Another way to understand pin spacing is to re-work Equation #6-12 which was:

$dR/dA = (-2V^2/g) \cos(2A)$, Equation #6-12.

The pin spacing, were one to ignore parallax, would relate to the reciprocal of that, or:

$$dA/dR = 1/((-2V^2/g) \cos(2A))$$
$$= g/2V^2 \cos(2A)$$

The "dA" change will be the angle defined by the pin spacing and peep-to-pin distance such that dh/l = tan dA. Thus:

$$dh = l \tan dR$$
$$= l \tan [(dR \times g)/2V^2 \cos(2A)]$$

Example:

$V = 180$ fps

$dR = 10$ yards

$A = 3°$ (per previous calc for about 35 yard range)

$\cos(2A) = \cos 6° = 0.9945$

$(dR \times g)/2V^2 \cos(2A) = 10$ yds $\times$ 32.2ft/sec^2/(2 $\times$ (180ft/sec)^2 $\times$ 0.9945

= 0.01499 radians = 0.859 degrees

tan dr = tan 0.859 degrees = 0.01499

$$dh = l \times 0.01499$$
$$= 28.75" \times 0.01499 = 0.431"$$

This method of pin space calculation gives the same result as the longer one, but it has the advantage of making obvious the parameters that matter in pin spacing. Look again at the formula:

$dh = l \tan [(dR \times g)/2V^2 \cos(2A)]$, Equation #14-3.

For the small angles used in archery, the tangent of an angle and the angle itself are pretty much the same thing. Thus:

$dh = (l \times dR \times g)/(2V^2 \cos(2A))$, Equation #14-4

The first term, "l" is peep-to-pin-plane distance. The longer "l" is, the larger the pin spacing. Some hunters put their pins on the near side of their bows to avoid snags; their pins will end up more closely spaced.

The second term, "dR" is the range between pins. Obviously, pins twenty yards apart are spaced twice as far as are pins ten yards apart.

The "g" (gravity) term is a constant. The "cos(2A)" term varies only a little with range.

The "V^2" term is the only big variable that isn't intuitively obvious. The formula says that pin spacing is inversely proportional to the square of arrow velocity!

Examples of 10-yard increment spacing for the dimensions of the author's bow are:

Table 14-1 ... PIN SPACING, FRICTIONLESS

Assumed data:
Peep-to-pin-plane distance = 28.75"

Arrow elevation angle = 3°

Spaces are for pins at 10-yard intervals.

Formula: $dh = \frac{l \times dR \times g}{2V^2 \cos 2A}$

140 fps .... 0.71"

160 fps .... 0.54"
180 fps .... 0.43"

200 fps .... 0.35"

220 fps .... 0.29"

These numbers are reasonably accurate for the real world of archery. The velocity to use is a weighted average of launch and arrival velocities, weighted: V,avg = 1/4 x (3 x V,launch + 1 x V, arrival).

Second, the range has to be beyond 30 yards, least parallax negate results.

PIN SPACING WITH FRICTION

Chapter 8, "Trajectories with Friction", delineated how to compute actual arrow elevation angles with friction. I used the same 2117 arrow with 5" 3-fletch with field point. It weighs 527 grains and has a drag of 386 grains at an initial velocity of 200 fps. Calculating the velocities at each 10-yard interval and the arrow elevation angles permits calculating the pin locations and thus the actual space between pins. Data is printed in table below. The "dh" column shows the data sought, the distance between pins.

Table 14-2 ... PIN SPACING, ACTUAL

Range (yds)	Velocity (fps)	Parallax (degrees)	Elevation (degrees)	(A + P) (deg)	h (")	dh (")
0	200.0	7.720	.000	7.720	3.897"	-
10	196.5	.572	.698	1.270	.637	-3.3"
20	193.0	.297	1.409	1.706	.856	.22
30	189.5	.201	2.133	2.334	1.172	.32
40	186.0	.151	2.870	3.021	1.517	.35
50	182.6	.122	3.622	3.744	1.881	.36
60	179.2	.102	4.389	4.491	2.258	.38
70	175.9	.087	5.171	5.258	2.646	.39
80	172.5	.076	5.969	6.045	3.045	.40
90	169.2	.068	6.784	6.852	3.455	.41
100	165.9	.061	7.616	7.677	3.876	.42

Pin Spacing Parameters

It is all very nice to do a computer print-out of a bunch of numbers, but they don't mean much unless some conclusions can be drawn. The question in my mind was, "What velocity does pin spacing correspond to: Final? Average? Terminal?" My assumption was that the pin spacing between the 80 and 90 yard pins, for instance, would correspond to the average arrow speed when traveling between 80 and 90 yards. To answer that question, I asked the computer to figure arrow elevation angles first with friction and then without friction. The procedure for calculating with friction is given in Chapter 8, "Trajectories With Friction". Without friction, the formula is:

$A = 0.5 \text{ arcsine } (Rg/V^2)$, Equation #6-10

However, when computing without friction, I asked the computer to use various velocities. In the table below, the first columns under "frictionless" show results using initial arrow velocity unchanged at all ranges. Not unexpectedly, it shows flatter shooting and with less change of arrow elevation for a given change of range than for shots with friction.

In the second set of columns, I used a weighted average velocity where the initial velocity has a weight of 3 and the final velocity has a weight of one. Why? Because that weighting gave better correlation than any other.

In the third set of columns, an average of launch velocity and arrival velocity was used.

In the fourth set of columns, the average velocity over the range between pins was used. For instance at 80 yards, the average of the velocity at 70 yards plus the velocity at 80 yards was used.

It was much to my surprise that the 3:1 weighting of launch velocity versus arrival velocity gave good correlation and that bad correlation came from using the interval's average velocity.

Table 14-3 ... ARROW ELEVATION ANGLES, WITH & WITHOUT FRICTION

Range (yds)	<.. With friction ..>			<......... Without Friction .... > $(A = 1/2 \text{ arcsine}(Rg/V^2))$							
	V (fps)	A (°)	dA (°)	V=Vo		3Vo+Vt		Vo+Vt		Vx+Vt	
				A	dA	A	dA	A	dA	A	dA
0	200.0	.000	N.A.	.000	N.A.	.000	N.A.	.000	N.A.	.000	N.A.
10	196.5	.698	.698	.692	.692	.698	.698	.704	.704	.704	.704
20	193.0	1.409	.711	1.384	.692	1.409	.711	1.434	.730	1.460	.756
30	189.5	2.133	.724	2.077	.693	2.133	.724	2.191	.757	2.272	.812
40	186.0	2.870	.737	2.772	.694	2.872	.739	2.977	.786	3.147	.874
50	182.6	3.622	.752	3.468	.696	3.624	.753	3.792	.815	4.088	.941
60	179.2	4.389	.767	4.166	.698	4.393	.768	4.639	.847	5.101	1.013
70	175.9	5.171	.782	4.866	.701	5.177	.784	5.518	.879	6.193	1.092
80	172.5	5.969	.798	5.570	.704	5.979	.803	6.436	.918	7.377	1.184
90	169.2	6.784	.815	6.277	.707	6.799	.820	7.391	.954	8.664	1.287
100	165.9	7.616	.832	6.988	.711	7.638	.840	8.387	.997	10.064	1.399
	< Actual data >					< Best fit>					

Summary of Pin Spacing Factors

The spacing between sighting pins depends upon:

1. Arrow speed.
2. Geometry of peep sight and sighting pins.
3. Arrow friction and weight.

It was quite a surprise to the author to learn that arrow friction and weight are very minor factors in pin spacing. They are no factor at all at short ranges.

The geometry of peep sight and sighting pins is the <u>only</u> factor at very short ranges and, for conventional archery velocities configurations, is very significant out to 30 yards.

Arrow launch velocity is the predominant factor determining pin spacing beyond 30 yards. The values given in Table 14-1 are reasonable accurate, particularly for the middle distances of 30, 40, 50 & 60 yards. They are computed from the equation:

$dh = (l \times dR \times g)/(2V^2 \cos 2A)$, Equation 14-4.

where:

dh = pin spacing (inches or other measure)

l = peep-to-pin-plane distance (inches or other measure)

dR = range increment (feet)

$g = 32.2 \text{ ft/sec}^2$.

V = launch velocity (ft/sec)

A = arrow elevation angle at launch

Sample Calculation of Velocity from Pin Spacing:

The author's Martin Cougar target bow has the following data; use it to compute arrow speed:

Peep-to-pin distance = 28.0"

Pin spacing: 10 yd to 20 yd = 0.300"

20 yd to 30 yd = 0.406"

30 yd to 50 yd = 0.938"

50 yd to 70 yd = 1.000"

Re-working Equation 14-4 to solve for velocity, knowing pin spacing yields:

$V = (l g dR/2dh \cos 2A)^{1/2}$ Equation 14-5.

where l = peep-to-pin distance = 28.0"

$g = \text{gravity} = 32.2 \text{ ft/sec}^2$

dR = difference in range between pins = 20 yards

dh = space between pins = 0.938"

$A = \frac{1}{2} \arcsin (Rg/V^2)$ where

R = average range between pins = 40 yards

V = arrow velocity & 1st aprox = 170 fps.

Note that a first approximation of arrow velocity has to be made in order to calculate arrow elevation angle. A large error in this approximation will make only a small difference in final result.

I've chosen to use the 30 yd to 50 yd pin spread because the minimum <u>must</u> be at least 30 yards in order <u>that</u> parallax may be ignored.

Simultaneously, the shortest range should be used in order that the effect of the slowing of the arrow during flight be minimized.

Sample calculation is:

$$A = 0.5 \arcsin\left[\frac{40 \times 3\text{ft} \times 32.2\frac{\text{ft}}{\text{sec}^2}}{(170\frac{\text{ft}}{\text{sec}})^2}\right]$$

A = 0.5 arcsin (3864/28900)
= 0.5 arcsin 0.1337 = 0.5 x 7.68°
= 3.84°.

$V = (lgdR/2dh \cos 2A)^{\frac{1}{2}}$
Equation 14-5.

$$V = \left[\frac{28'' \times 32.2\frac{\text{ft}}{\text{sec}^2} \times 20\text{yd} \times 3\frac{\text{ft}}{\text{yd}}}{2 \times 0.938'' \cos 2 \times 3.84^\circ}\right]^{\frac{1}{2}}$$

$$V = \left[\frac{54{,}096''\text{ft}^2/\text{sec}^2}{1.876'' \cos 7.68^\circ}\right]^{\frac{1}{2}}$$

$V = [54{,}096/1.859]^{\frac{1}{2}}$ ft/sec

V = 170.6 ft/sec.

What happens when shooting uphill or downhill? A number of things.

Velocity:

If shooting uphill, the arrow will arrive at a slower speed. How much slower? That depends strictly upon how much higher the target is. Arrow velocity and target elevation are inter-related as follows:

$$V_f^2/2g = V_o^2/2g - H.$$

where V_o = original velocity &
V_f = final velocity.
H = target elevation compared to launch elevation

If the target is at the same elevation as the archer, arrow should arrive at same speed at launched, assuming friction is ignored. If the target is lower, speed will be greater. If the target is higher, speed will be slower. Working the above formula to yield final velocity yields:

$$V_f = (V_o - 2gH)^{1/2},$$
Equation #15-1

Note that formula cannot apply when "H" is higher than the arrow can go. The highest an arrow can be shot was given in Chapter #6 as:

$$H_{max} = V_o^2/2g,$$
Equation #6-3.

Note also that these formulas ignore friction. It was shown in the chapter on friction that an arrow shot straight down is typically going roughly at "terminal" velocity. In other words, a typical arrow shot straight down will neither gain nor lose much speed. Thus any attempt to use the above formula to predict arrival speeds when the target is significantly lower than the launch site will be inaccurate. On the other hand, it was shown in the chapter on "Trajectories with Friction" that the use of launch velocity to predict pin sight spacing was quite good. Thus I conclude that the mathematical analysis of uphill shooting can ignore friction without much error. Downhill shooting analysis in this chapter will ignore friction even though friction errors may be rather significant.

Pin to Use:

Initially, in this chapter, parallax will be ignored. The questions to be answered are similar to the ones field archers pose to one another upon sighting an uphill or downhill target. The question are, "How far is the target, and what sighting pin should be used?" The "How far is the target?" is interpreted as the direct slant range as measured by a taut tape or a range finder. The "What sight pin should be used?" part is what this chapter will strive to answer. The author's field experience is that each archer has a different answer for the range pin to use, even though all are shooting at the same target. The calculations will confirm that, due to differences in arrow velocity, differences in arrow friction and weight, and differences in parallax, archers use different pins to shoot at the same target. The mathematical analysis will initially ignore friction, arrow weight, and parallax.

Trajectory Formula

The formula which locates every position along a frictionless arrow's trajectory is:

$y = x \tan A_H - gx^2/2V^2 \cos^2 A_H$,
Eqn 15-1 ... where:
y = elevation with respect to launch elevation.
x = horizontal distance from launch site.
g = gravity = 32.2 ft/sec^2
V = velocity of arrow at moment of launch
A_H = Angle of arrow launch above or below horizontal.

See Figure 15-1 ... Uphill/Downhill Geometry.

Equation 15-1 is used to compute exact locations along an arrow's trajectory when the launch elevation angle is known.

Launch Angle

By manipulating Equation 15-1, a formula for calculating the needed launch angle can be derived. The formula is:

$A_H = \arctan [N - (N^2 - 2N \tan A_E - 1)^{1/2}]$ ... Eqn. 15-2 ... where:
$N = V^2/(gR \cos A_E)$.

A_H = Launch angle above or below horizontal.
A_E = Elevation angle of target above or below horizontal.
g = gravity = 32.2 ft/sec^2.
R = range, slant, from launch direct to target.

See Figure 15-1 ... Uphill/Downhill Geometry.

There are technically two angles which will get the arrow to the target, one being the traditional flat trajectory and the other being a high lobbing shot. By adding rather than subtracting the square root function in Equation 15-2, the high shot can be computed.

The launch angle compared to the line-of-sight is the angle above horizontal less the target elevation angle, given by:

$A_S = A_H - A_E$.
Equation 15-3.

Equivalent Horizontal Range

The question we set out to answer was, in effect, "What range would an arrow get if it were shot at a horizontal target at an angle above the horizontal equal to the angle above the line of sight? In Equation 15-3, we calculated the angle above line-of-sight for a uphill or downhill shot. Now the question is, if that same angle were used above the horizontal, how far would the arrow travel horizontally. The answer can be computed by using the formula given in Chapter 6, "Trajectories", for horizontal frictionless shots:

$A = \frac{1}{2} \arcsin (Rg/V^2)$
Equation #6-11.

Setting A equal to A_S yields:

$A_S = \frac{1}{2} \arcsin (RG/V^2)$.
Rearranging yields:
$\arcsin (RG/V^2) = 2A_S$.
Rearranging yields:
$RG/V^2 = \sin(2A_S)$.
Rearranging yields:

$R_E = (V^2/G)\sin(2A_S)$
Equation #15-4, where

R_E = range, equivalent horizontal.
V = velocity of arrow at launch.
G = gravity, 32.2 ft/sec^2.
A_S = angle above line-of-sight of launch to uphill or downhill target.

Sample Calculation

Assumed data: V = 200 ft/sec.
R = 60 yds = 180 ft. $A_E = -40°$.

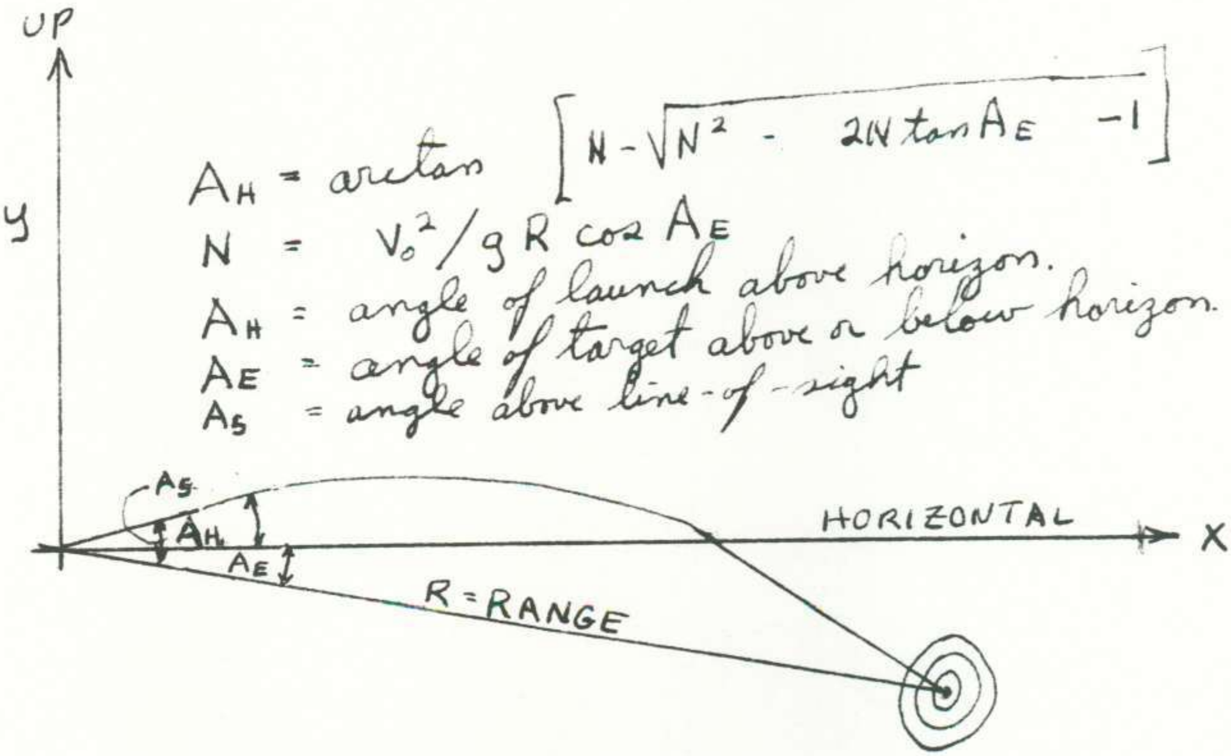


Figure 15-1 ... UPHILL/DOWNHILL GEOMETRY

$N = V^2/(Rg \cos A_E) =$
$(200 ft/sec)^2/(180 ft/sec \times 32.2 ft/sec^2 \times \cos -40°)$
$N = 9.0090$
$A_H = \arctan[N-(N^2 - 2N \tan -40° - 1)^{1/2}] = \arctan[N-(N^2 + 1.6782N - 1)^{1/2}]$
$= \arctan[9.009 - (9.009^2 + 1.6782 \times 9.0090 - 1)^{1/2}]$
$= \arctan[9.009 - (81.162 + 15.119 - 1)^{1/2}] = \arctan[9.009 - 95.281^{1/2}]$
$= \arctan[9.009 - 9.7612] = \arctan -0.7522$
$= -36.95°$ = launch angle below horizon shooting at a target 40° down.

$A_S = A_H - A_E$
$= -36.95° - (-40°) = + 3.05°$
= launch angle above line-of-sight.

$R_E = (V^2/G) \text{ sine } 2A_S$
$= \frac{(200 ft/sec)^2}{32.2 ft/sec} \sin (2 \times 3.05°)$
$= 1{,}242 \text{ ft sine } 6.10°$
$= 1242 \text{ ft} \times 0.1063 = 132 \text{ ft}$
= 44.0 yards = equivalent horizontal range.

Conclusion: For a 200 fps arrow shooting 40° down at a range of 60 yards, shot should be made as though shooting from 44 yards. Keep in mind that parallax has been ignored. Since the parallax at 44 yards is different than at 60 yards, some error exists. Beyond a range of 30 yards, parallax is fairly small and the differences between parallaxes is very small.

<u>Tabulation of Data</u>:

The formulas and procedures for computing equivalent ranges for uphill and downhill shots given in this chapter shed very little light upon the parameters which influence the equivalent ranges. The only thing to do, then, is to tabulate a large number of variables and see how the results vary.

Table 15-1 ... 60 yards @ 150 fps:

Target angle (deg)	A_H (deg)	A_S (deg)	R_E (yds)	(%)
89	89.15	.15	1.2	2.1
45	50.86	5.86	47.3	78.8
30	36.94	6.94	55.9	93.2
15	22.47	7.47	60.1	100.1
10	17.53	7.53	60.5	100.8
5	12.52	7.52	60.5	100.8
0	7.46	7.46	60.0	100.0
-5	2.35	7.35	59.1	98.5
-10	-2.82	7.18	57.8	96.3
-15	-8.03	6.97	56.1	93.5
-20	-13.29	6.71	54.1	90.1
-25	-18.59	6.41	51.7	86.1
-30	-23.93	6.07	49.0	81.6
-40	-34.72	5.28	42.7	71.1
-60	-56.64	3.36	27.2	45.4
-70	-67.72	2.28	18.5	30.8
-89	-88.88	.12	.9	1.6

Table 15-2 ... 60 yards @ 250 fps:

Target angle (deg)	A_H (deg)	A_S (deg)	R_E (yds)	(%)
89	89.05	.05	1.1	1.8
45	46.95	1.95	43.9	73.2
30	32.36	2.36	53.2	88.7
15	17.60	2.60	58.7	97.8
10	12.64	2.64	59.6	99.3
5	7.66	2.66	60.0	100.0
0	2.66	2.66	60.0	100.0
-5	-2.36	2.64	59.5	99.2
-10	-7.40	2.60	58.6	97.7
-15	-12.46	2.54	57.3	95.5
-20	-17.54	2.46	55.5	92.5
-25	-22.64	2.36	53.4	88.9
-30	-27.75	2.25	50.8	84.7
-40	-38.02	1.98	44.7	74.4
-60	-58.72	1.28	28.9	48.1
-70	-69.13	.87	19.7	32.8
-89	-88.96	.04	1.0	1.7

Table 15-3 ... Recapitulation of Equivalent Horizontal Ranges for Slant Ranges of 30, 60 & 120 yards @ 150, 200 & 250 fps

Range = 30 yards

Velocity = Elevation	150 (fps)	200 (fps)	250 (fps)
45°	22 yds	22 yds	22 yds
30°	27	27	26
15°	30	29	29
0°	30	30	30
-15°	29	29	29
-30°	25	26	26
-40°	22	23	23

Range = 60 yards

Velocity = Elevation	150 (fps)	200 (fps)	250 (fps)
45°	47 yds	45 yds	44 yds
30°	56	54	53
15°	60	59	59
0°	60	60	60
-15°	56	57	57
-30°	49	50	51
-40°	43	44	45

Range = 120 yards

Velocity = Elevation	150 (fps)	200 (fps)	250 (fps)
45°	116 yds	96 yds	91 yds
30°	125	113	109
15°	126	121	119
0°	120	120	120
-15°	109	112	113
-30°	93	97	100
-40°	80	85	87

Conclusions:

Three archers standing side-by-side shooting at the same target will need to use different range pins. For a target uphill 45° at 60 yards, for instance, the pins to use will be 47 yards, 45 yards and 44 yards respectively for archers launching at 150 fps, 200 fps and 250 fps. See Table 15-3 above.

The pin to use is almost always a shorter range pin. The only exceptions are when the target is slightly uphill. See Table 15-1 for 60 yards at 150 fps. The exception is so small that it can be taken as a rule both uphill and downhill shots use the actual or shorter range pins.

Thumb Rules

1. If target is between +15° and -10°, make no correction.

2. For targets higher than +15° or lower than -10°, shoot as though target were closer.

3. For downhill shots, the amount of correction needed increases with range.

Dimensionless Parameters

The preceeding mathematical analysis is frustrating in that there is no clear reason why only a slightly uphill shot should use a longer sighting pin. Worse, there is no clue as to why some slightly uphill shots don't use longer pins. Academic type engineers have a great fondness for dimensionless numbers, such as Reynold's numbers, Mach numbers, Prandtl numbers, Nusselt numbers, etc. Searching for a similar approach, I decided to express range as a dimensionless ratio of range divided by maximum horizontal range. The answers come out as dimensionless pin-to-use range divided by slant range.

Equations 15-2 & 15-4 can be generalized by noting the the maximum horizontal range is:

$R_{max} = V^2/g$, Equation #6-4.

Substituting yields:

$N = V^2/(gR \cos A_E)$
$= R_{max}/(R \cos A_E)$. Using this in Eqn. 15-3 yields:

$A_H = \arctan[(R_{max}/R \cos A_E - ((R_{max}/R \cos A_E)^2 - 2R_{max} \tan A_E/R \cos A_E - 1)^{½}]$ Equation 15-5.

Rearranging Equation #15-4 and dividing both sides by R yields:

$R_E/R = (R_{max}/R) \text{ sine } (2(A_H - A_E))$ Equation 15-6.

With these two formulas, we can look at which pin to use without looking at arrow velocity nor specific ranges. Rather, the ratio of maximum range to slant range is one input parameter. The target's elevation angle is the second input parameter. The ratio of pin-to-use versus slant range is the output parameter. The resulting output is applicable to thrown rocks, artillery shells, rifle shots, as well as archery shots.

Table 15-4 ... Ratios of Pins-to-Use versus Slant Ranges

TARGET ELEVATION ANGLE	R/R,m .05	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
89°	.018	.018	.020	.021	.024	.035	-	-	-	-	-
75°	.265	.273	.290	.314	.352	.452	-	-	-	-	-
60°	.511	.524	.553	.592	.648	.759	-	OUT OF RANGE			-
45°	.720	.734	.766	.806	.859	.944	-	-	-	-	-
40°	.779	.792	.823	.861	.910	.983	1.162	-	-	-	-
35°	.831	.844	.873	.907	.951	1.011	1.125	-	-	-	-
30°	.877	.889	.915	.945	.982	1.031	1.111	-	-	-	-
15°	.972	.979	.993	1.008	1.025	1.045	1.070	1.107	-	-	-
5°	.998	1.001	1.005	1.010	1.015	1.021	1.027	1.035	1.046	1.066	-
0°	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-15°	.960	.954	.942	.930	.919	.908	.896	.884	.871	.858	.842
-30°	.855	.845	.826	.809	.792	.776	.760	.745	.730	.715	.701
-45°	.695	.684	.663	.644	.627	.611	.597	.583	.570	.557	.545
-60°	.490	.480	.463	.448	.434	.422	.411	.401	.391	.382	.374
-75°	.253	.247	.238	.229	.222	.215	.209	.204	.199	.195	.190
-89°	.017	.017	.016	.015	.015	.014	.014	.014	.013	.013	.013

Typical archery shots range from 0.05 to 0.50 in above table, depending upon the bow used. For instance the R/R_{max} ratios used in Table 15-3 amount to:

Velocity	R_{max}	30 yards	60 yards	120 yards
150 fps	233 yds	0.13	0.26	0.52
200 fps	414 yds	0.07	0.15	0.29
250 fps	647 yds	0.05	0.09	0.19

I am personally most frustrated that a clear description of when to shoot long and when to shoot short is not obvious. The difficulty can be seen by studying the target which is uphill 30°. At short range, it is to be shot as though the range were only 87.7% of what it is. At long range, it is to be shot as though the range were 111.1%! My recommendation is to prepare a table for your own bow's speed and for typical ranges. The author's 74# compound hunting bow shooting a 30" 2117 hunting arrow has a velocity of about 200 fps. Using the data in tables in this chapter, the following table for my bow's particular arrow velocity applies:

Table 15-5...PINS TO USE FOR 200 FPS ARROW

TARGET ELEVATION	RANGE: 30 YARDS PIN	(%)	60 YARDS PIN	(%)	120 YARDS PIN	(%)
+45°	22	(73)	45	(75)	96	(80)
+30°	27	(90)	54	(90)	113	(94)
+15°	29	(97)	59	(98)	121	(101)
0°	30	(100)	60	(100)	120	(100)
-15°	29	(97)	57	(95)	112	(93)
-30°	26	(87)	50	(83)	97	(81)
-40°	23	(77)	44	(73)	85	(71)

ACCURACY STATISTICS

The shape and size of groups of arrows shot in a hunting environment is what this chapter will compute.

Four elements are involved:

1. Launch accuracy.
2. Range estimation accuracy.
3. Range.
4. Whether range is estimated short or long.

To do this, statistical analysis of actual data will be done. The data analyzed consists of the author's own shooting results at 30-yards plus the author's own range estimation results at various ranges from 8 to 100 yards.

A number of facts and/or assumptions need to be stated:

First, launch data should show that shots tend to hit the center of the target. If they actually group high, low, left or right, the archer simply has his pins set wrong. It would be possible, of course, that the archer has a flaw which, for instance, has his good shots centering on target but his bad shots favoring right but not left. My own shots don't seem to show any such tendency. Thus I recorded only how far from center each hit was, but did not record each shot's left/right nor up/down data.

Second, range estimation data might show an aggregate tendency to be long or short. My own data showed an amazing lack of that. The sum of all range estimates was almost exactly equal to the sum of all ranges estimated. Thus, for the author at least, the data can be processed with the assumption that under-estimates and over-estimates are equally likely.

Third, the absolute accuracy of a ranges estimated is obviously better at close range than at long range. What is not obvious but which seems to be true of my range estimates is that my accuracy when expressed as a percent of the range seems to be constant. Specifically, my accuracy averages 11% of range. Thus at 20 yards my average error is 11% of 20 yards or +/-2.2 yards and at 100 yards it is +/-11 yards. Accuracy being a constant percent of range estimated seems logical and thus is assumed to be true.

The hit patterns to be expected in a hunting environment should be described before presenting the data and the statistical manipulation of it. The unique feature of the hunting environment is that the range can be known by one method only: estimation. This is different than target archery, obviously. It is also different than shooting on field ranges, as archers tend to memorize field range distances. Also, more than one shot is usually taken on field ranges. The author's second shot on a field range is as accurate as on a target range. The author's first shot at a field target depends upon how well he remembers from previous use of the range how far the target is.

The accuracy of hunting shots taken at close range, particularly those within the range of the 10-yard pin, should be as accurate as target archery. Thus the pattern of hits should be round at close range, same as in target archery.

The patterns of hunting shots taken at long range should show the same left/right dispersion as target shots. But the high/low dispersion should be much greater due to the errors in estimation of range.

Data Analyzed

I shot 100 arrows from 30 yards to ascertain what to expect at a known range. Second, I estimated 76 random ranges to ascertain the accuracy and consistency to expect in range estimation. Third, I mathematically combined the two separate elements.

Fixed Range Shooting:

The 100 arrows shot from 30 yards were 2018s, 30" long, with 5" 3-fletch. Bow was a 55# Martin Cougar compound with pins and peep sight. The results are tabulated below.

Table 16-1 ... Tabulation of Arrow Hit Data at 30 Yards

RADIUS (inch)	ARROW HITS	AREA (in^2)	DENSITY (hits/in^2)	log(H/A)
.0	2	.20	10.186	1.0080
.5	10	1.57	6.366	.8039
1.0	9	3.14	2.865	.4571
1.5	11	4.71	2.334	.3682
2.0	16	6.28	2.546	.4059
2.5	16	7.85	2.037	.3090
3.0	14	9.42	1.485	.1719
3.5	6	11.00	.546	-.2631
4.0	7	12.57	.557	-.2541
4.5	2	14.14	.141	-.8493
5.0	4	15.71	.255	-.5941
5.5	2	17.28	.116	-.9365
6.0	0	18.85	.000	
6.5	0	20.42	.000	
7.0	1	21.99	.045	-1.3422
Total:	100	165 in^2		

Finding a mathematically manipulatable description of the data takes some doing. In the left column is the miss distance in 1/2" increments. "1.0" includes all hits from 3/4" to 1¼" from target center to arrow center. Inspecting the "hits" column, it is clear that the majority of arrows were from 1/4" to 3¼" from dead center. Looking only at the hits versus distance-from-center, it would appear that one of the safer places for an insect on the target face to be would be at the exact center. That doesn't make sense. The answer is to note that the area at each radius is larger farther from center, as shown in the third column. Dividing hits by area yields a more meaningful parameter, namely hits per square inch versus distance from center of target. Doing so gives a logical result, namely that the most dangerous place for an insect to be is at the center of the target. Plotting the hits/in^2 versus radius yields a very steep non-linear curve. Non-linear curves are not amenable to statistical analysis, so I searched for a manipulation which would yield a linear representation. The logarithm of the hit density gave excellent linearity.

Standard deviations:

Statisticians have worked up mathematical formulas to describe the probabilities of random events happening. In the case of target archery, the "event" is that of the arrow hitting any particular distance away from dead center. They have manipulated that math in such a way as to satisfy themselves that 1 standard deviation includes 68¼% of all events; 2 standard deviations include 95.45%; and 3 standard deviations include 99.73%.

The formulas used and the calculations themselves are shown in Figure 16-1.

Figure 16-1 ... Shooting Accuracy Statistical Calculations

Data = author's results at 30 yards with 55# compound shooting 30" 2018's having 5" 3-fletch vanes and 125-grain field points.

	MISS (x)	HITS	AREA (sq.in.)	H/A (hits/in²)	log(H/A) (y)	(x^2)	(y^2)	(xy)
	.0	2	.20	10.186	1.0080	.0	1.016	.000
	.5	10	1.57	6.366	.8039	.2	.646	.402
	1.0	9	3.14	2.865	.4571	1.0	.209	.457
	1.5	11	4.71	2.334	.3682	2.3	.136	.552
	2.0	16	6.28	2.546	.4059	4.0	.165	.812
	2.5	16	7.85	2.037	.3090	6.3	.095	.773
	3.0	14	9.42	1.485	.1719	9.0	.030	.516
	3.5	6	11.00	.546	-.2631	12.2	.069	-.921
	4.0	7	12.57	.557	-.2541	16.0	.065	-1.016
	4.5	2	14.14	.141	-.8493	20.2	.721	-3.822
	5.0	4	15.71	.255	-.5941	25.0	.353	-2.970
	5.5	2	17.28	.116	-.9365	30.2	.877	-5.151
	7.0	1	61.26	.016	-1.7872	49.0	3.194	-12.51
Totals:	40.0	100	165.13		-1.1603	175.5	7.576	-22.88
Averages:	3.08	7.69	12.70	2.265	-.0893	13.5	.583	-1.760

$N = 13 \quad \Sigma x = 40 \quad \bar{x} = 3.08 \quad \Sigma x^2 = 175.5$

$\Sigma xy = -22.88 \quad \Sigma y = -1.1603 \quad \bar{y} = -0.0893 \quad \Sigma y^2 = 7.576$

$$\sigma_y = \left[\frac{\Sigma y^2 - \frac{(\Sigma y)^2}{N}}{N-1}\right]^{\frac{1}{2}} = \left[\frac{7.576 - \frac{(1.1603)^2}{13}}{13-1}\right]^{\frac{1}{2}} = 0.789$$

$$\sigma_x = \left[\frac{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}{N-1}\right]^{\frac{1}{2}} = \left[\frac{175.5 - \frac{40^2}{13}}{13-1}\right]^{\frac{1}{2}} = 2.090$$

$$m = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma x^2 - \frac{(\Sigma x)^2}{N}} = \frac{-22.88 - \frac{40(-1.1603)}{13}}{175.5 - \frac{40^2}{13}} = -0.368$$

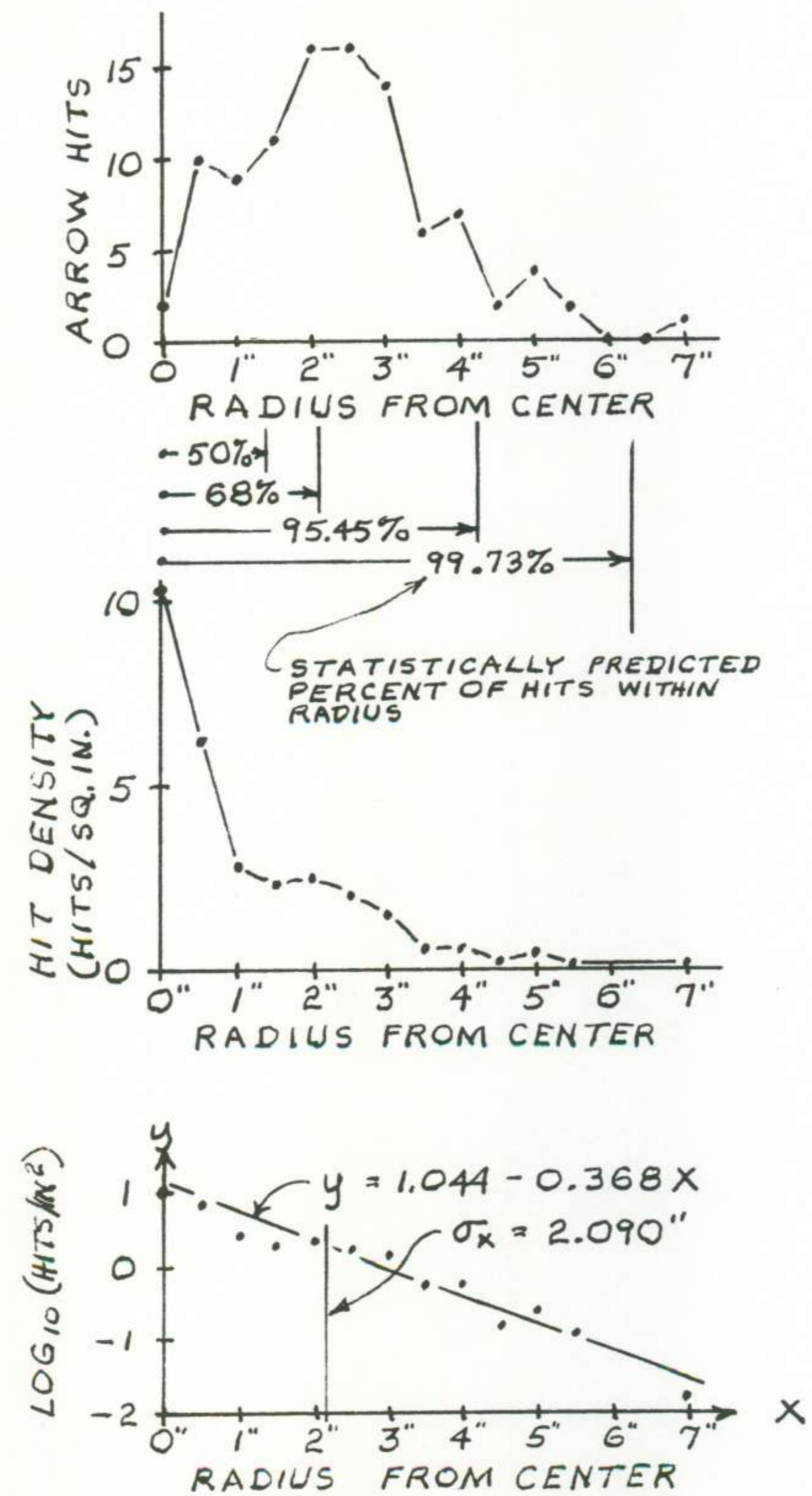
$$y\text{-intercept} = b = \frac{\Sigma y - m\Sigma x}{N} = \frac{-1.1603 - (-0.368 \times 40)}{13} = 1.0441$$

$R = m\,\sigma_x / \sigma_y = -0.368 \times 2.090 / 0.789 = -0.975$ = correlation

$y = 1.0441 - 0.368\,X = \log_{10}(\text{HITS}/\text{IN}^2)$

$X_{\sigma=1.0}$: $1.0441 - 0.789 = 0.2551 \rightarrow 1.80$ hits/in² @ 2.09"

$X_{\sigma=2}$: $1.044 - 2 \times 0.789 = -0.534 \rightarrow 0.29$ hits/in² @ 4.18"



Figures 16-1 ... Plots of Target Hit Data

Upon plotting the above data for my 100 shots at 30 yards, I found:

Standard deviations	Radius	Diameter	Percent of shots	Launch error
0.674	1.41"	2.82"	50.00%	1.30 mils
1.00	2.09"	4.18"	68.25%	1.94 mils
2.00	4.18"	8.36"	95.45%	3.87 mils
3.00	6.27"	12.34"	99.73%	5.80 mils

I was surprised that there turned out to be a linear relationship between standard deviations and launch error.

Launching accuracy

Launching accuracy has nothing to do with target range. If the arrow is shot one degree left of target, that's the error: 1° left. The resulting miss will, of course, depend upon range. If the range is 100 yards, the miss will be 10 times (at least) as much as from 10 yards.

"Accuracy", from a statistician's point of view, is a probability term. An archer of known accuracy has a known probability of placing an arrow a particular distance from dead center. The author is not aware of any convention for describing archery accuracy, so I've adopted the artillery term of "mils" to describe errors. A "mil" is a miss equal to one one-thousandth of the range. One mil at 30 yards would be:

$$1 \text{ mil} = \frac{30 \text{ yds} \times 36 \text{ in/yd}}{1{,}000} = 1.08 \text{ inches @ } 30 \text{ yd.}$$

One might have chosen degrees or radians, but mils work out to be easy numbers to work with and understand.

Definition of an archer's accuracy:

On the basis of the above data, I consider myself to be a "3.87 mil" archer, as I launched 95% of all shots with an accuracy of 3.87 mils or better. When archers say they can put all their shots within a group of a particular size they usually mean that they can put about 95% within a given diameter. Since 95.45% is what the mathematicians call "two standard deviations", this seems to me like a good definition. Without going through all the math, it would be easiest to judge your own accuracy against the 1.0 standard deviation criteria, which is 68½%. In other words, shoot 100 shots. Find the radius which divides the inner 68 shots from the outer 32 shots. Convert that radius to mils per above example. Double those mils to get your "2-standard deviation accuracy" &/or your "95½% accuracy".

Range estimation accuracy:

The accuracy of eye-ball range estimation varies, on average, from about 30% of range to 10% range. To ascertain your own accuracy, take a range-finder and practice estimating range. For a simple calculation of your average accuracy, simply write down the estimated range, the actual range, the difference, the difference divided by the actual range and multiplied by 100. Then take the absolute average by adding up all the percent errors while ignoring the +/- signs and dividing by the number of estimates. To ascertain how long or short your average estimate is, add all the estimated and actual ranges and divide one by the other.

To predict the patterns to expect in a hunting environment, I needed a statistical analysis of my actual results. The data is shown in Table 16-1. The standard deviation is computed using the following formula:

$$SD = [((sum(X^2) - (sum\ X)^2/N)/(N-1)]^{1/2}$$

where:

SD = standard deviation
X = error expressed as percent-of-range-estimated
N = number of estimates

Table 16.1 ... Range Estimation Data

GUESS (yds)	ACTUAL (yds)	<.... ERROR (yds)	(%)	....> ($\%^2$)
27	30	-3	-10.0	100
28	29	-1	-3.4	12
63	60	3	5.0	25
58	52	6	11.5	133
24	30	-6	-20.0	400
48	39	9	23.1	533
80	72	8	11.1	123
55	53	2	3.8	14
30	34	-4	-11.8	138
27	31	-4	-12.9	166
20	24	-4	-16.7	278
38	30	8	26.7	711
75	65	10	15.4	237
40	31	9	29.0	843
22	20	3	12.8	164
33	44	-11	-25.0	625
50	55	-5	-9.1	83
45	39	6	15.4	237
58	54	4	7.4	55
50	45	5	11.1	123
80	120	-40	-33.3	1,111
38	44	-6	-13.6	186
38	41	-3	-7.3	54
60	60	0	.0	0
50	45	5	11.1	123
65	62	3	4.8	23
23	23	0	.0	0
70	62	8	12.9	166
52	48	4	8.3	69
40	33	7	21.2	450
18	20	-2	-10.0	100
65	57	8	14.0	197
48	53	-5	-9.4	89
40	39	1	2.6	7
38	48	-10	-20.8	434
45	47	-2	-4.3	18
40	38	2	5.3	28
27	27	0	.0	0
43	42	1	2.4	6
75	80	-5	-6.3	39

GUESS (yds)	ACTUAL (yds)	<.... ERROR (yds)	(%)	....> ($\%^2$)
35	40	-5	-12.5	156
50	45	5	11.1	123
48	39	9	23.1	533
70	70	0	.0	0
75	75	0	.0	0
32	42	-10	-23.8	567
8	11	-3	-27.3	744
38	44	-6	-13.6	186
40	45	-5	-11.1	123
30	27	3	11.1	123
40	38	2	5.3	28
42	40	2	5.0	25
43	40	3	7.5	56
38	35	3	8.6	73
85	63	22	34.9	1,219
50	57	-7	-12.3	151
22	33	-11	-33.3	1,111
50	62	-12	-19.4	375
45	52	-7	-13.5	181
40	30	10	33.3	1,111
40	58	-18	-31.0	963
25	25	0	.0	0
26	30	-4	-13.3	178
28	23	5	21.7	473
40	41	-1	-2.4	6
50	40	10	25.0	625
27	23	4	17.4	302
55	60	-5	-8.3	69
14	17	-3	-17.6	311
10	9	1	11.1	123
13	13	0	.0	0
Averages: 43.1	43.2	-.11	.42	254
Totals: 3,272	3,281	-9	32.0	19,314

0.674 std dev: 10.81%
Mean: 11.10%
Avg. of absolutes: 13.04%
1.000 std dev = 16.04%
2.000 std dev = 32.08%
Number of estimates = 76

Hunting Group Calculation

All the information needed to calculate what the author's hunting group should look like at various ranges is now available. The raw data is:

Arrow velocity = 180 ft/sec.
Launch accuracy = 2.0 mils at 1.0 standard deviations.
Range estimation accuracy = 16.04% of range at 1.0 standard deviations.

The formulas needed are:

$M_{high} = E_R$ sine $A_{R=long}$, Eqn #11-1
where

M_{high} = miss distance when estimated range is longer than actual.
E_R = error in range estimate
$A_{R=long}$ = Angle of launch to hit target at estimated range.

$M_{low} = E_R$ sine $A_{R=R}$, Eqn #11-2,
where

M_{low} = miss distance when estimated range is shorter than actual.
E_R = error in range estimate
$A_{R=R}$ = Angle of launch to hit target at actual range.

In addition we need to know that the following is true of standard deviations:

0.674 includes 50.00% of all events.
1.000 includes 68.25% of all events.
2.000 includes 95.45% of all events.

The shape of groups for 20, 40, 60, and 80 yards will be computed at 0.674, 1.000 & 2.000 standard deviations. First, the left/right miss distances are computed from the 1.00 standard deviation = 1.94 mils results. Example: Miss at 2.00 std deviations at 80 yards = 80 yds x 36"/yd x 1.94 / 1,000 = 5.17".

Std. dev.	Mils	Left/right miss distances 20 yds	40 yds	60 yds	80 yds
0.674	1.30	0.94"	1.88"	2.82"	3.76"
1.000	1.93	1.39"	2.79"	4.18"	5.57"
2.000	3.87	2.79"	5.57"	8.36"	11.15"

Note that all are <u>radii</u>, not diameters. Note that all are lineal relationships: Double the range; double the error. Double the standard deviations; double the error. Note that arrow velocity does not enter the computation.

Calculation of Low Misses:

Example: Arrow velocity = 180 ft/sec.

Std. dev.	Percent of Range	Error in range estimate 20 yds	40 yds	60 yds	80 yds	
0.674	10.81%	2.16	4.32	6.49	8.65	= yds error.
"	"	17.84	35.68	53.51	71.35	= yds estimated.
"	"	2.3"	9.3"	21.0"	37.4"	= inches low due range.
"	"	0.9"	1.9"	2.8"	3.8"	= inches due launch.
"	"	3.2"	11.2"	23.8"	41.2"	= total miss low.
1.000	**16.04%**	3.21	6.42	9.62	12.83	= yds error.
"	"	16.79	33.58	50.38	67.17	= yds estimated.
"	"	3.4"	13.8"	31.1"	**55.5"**	= low due range error.
		1.4"	2.8"	4.2"	5.6"	= low due launch.
		4.8"	16.6"	35.3"	61.1"	= total miss low.

Std. dev.	Mils	Left/right miss distances 20 yds	40 yds	60 yds	80 yds
0.674	1.30	0.94"	1.88"	2.82"	3.76"
1.000	**1.93**	1.39"	2.79"	4.18"	5.57"
2.000	3.87	2.79"	5.57"	8.36"	11.15"

2.000	32.08%	6.42	12.83	19.25	25.66	= yards error.
"	"	13.58	27.17	48.75	54.34	= yards estimated.
"	"	6.9"	27.6"	62.2"	111.0"	= low due range error.
"	"	2.8"	5.6"	8.4"	11.2"	= low due launch.
"	"	9.7"	33.2"	70.6"	122.2"	= total miss low.

Correct launch angle = 1.709° 3.425° 5.153° 6.900°

The calculation of error in estimate is simply range multiplied by percent of range. The estimated range is then the actual less the error. Next the angle of launch which should be used for the actual range is computed. Example:

$$A_R = \tfrac{1}{2}\arcsin(Rg/V^2) = \tfrac{1}{2}\arcsin(80\text{yd} \times 3\text{ft/yd} \times 32.2\text{ft/sec}^2)/(180\text{ft/sec})^2$$
$$= \tfrac{1}{2}\arcsin(0.239) = \tfrac{1}{2} \times 13.8° = 6.900° \text{ (for 80 yards).}$$

Next the amount of the miss is computed (for 80 yards at 1.0 standard deviation) using:

$$M_{low} = E_R \text{ sine } A_{R=R} \quad \ldots \quad \text{Eqn \#11-2,}$$
$$= 12.83 \text{ yards sine } 6.900° = 12.83 \text{ yds} \times 36''/\text{yd} \times .120$$
$$= 55.5'' \text{ low.}$$

Finally, the launch error for the same standard deviations are added to the error caused by range estimate error.

Calculation of High Misses:

High misses are calculated differently because the formula for calculating the miss distance uses the (incorrectly) estimated range rather than the actual range. Otherwise the procedure is the same. Data is arranged as follows:

Std. dev.	Percent of Range	Error in range estimate 20 yds	40 yds	60 yds	80 yds	
0.674	10.81%	2.16	4.32	6.49	8.65	= yds error.
"	"	22.16	44.32	66.49	88.65	= yds estimated.
"	"	1.89°	3.80°	5.72°	7.66°	= actual launch angle.
"	"	2.6"	10.3"	23.3"	41.5"	= high due range.
"	"	0.9"	1.9"	2.8"	3.8"	= inches due launch.
"	"	3.5"	12.2"	26.1"	45.3"	= total miss high.
1.000	**16.04%**	3.21	6.42	9.62	12.83	= yds error.
"	"	23.21	46.42	69.62	92.83	= yds estimated.
"	"	1.98°	3.98°	5.99°	8.03°	= actual launch angle.
"	"	4.0"	16.0"	36.1"	**64.6"**	= high due range.
"	"	1.4"	2.8"	4.2"	5.6"	= high due launch.
"	"	5.4"	18.8"	40.3"	70.2"	= total miss high.
2.000	32.08%	6.42	12.83	19.25	25.66	= yards error.
"	"	26.42	52.83	79.25	105.66	= yards estimated.
"	"	2.26°	4.53°	6.83°	9.12°	= actual launch angle.
"	"	9.1"	36.5"	82.5"	146.4"	= high due range error.
		2.8"	5.6"	8.4"	11.1"	= high due launch.
		11.9"	42.1"	90.9"	157.5"	= total miss high.

Compiling the data computed above results in the data tabulated in the table which follows. Note that "Left" is entered but "Right" is omitted, as both are identical.

Table 16-2 ... Miss Distances for Hunting Environment Groups

Std	20 yards			40 yards			60 yards			80 yards		
Dev	Left	High	Low	Left	High	Low	Left	High	Low	Left	High	Low
.674	0.9"	3.5"	3.3"	1.9"	12.2"	11.2"	2.8"	26.1"	23.9"	3.8"	45.3"	41.3"
1.00	1.4"	5.4"	4.8"	2.8"	18.8"	16.7"	4.2"	40.3"	35.4"	5.6"	70.2"	61.3"
2.00	2.8"	11.9"	9.8"	5.6"	42.1"	33.4"	8.4"	90.9"	70.8"	11.1"	158"	123"

The above data is plotted in Figure 16-2. There are a number of extraordinary features shown in Figure 16-2, such as:

1) The groups have an amazing ratio of vertical size to width. The ratio gets bigger as range gets longer. Specifically, the ratios are about:

20 yds: 3.6 to 1. 40 yds: 6.3 to 1.
60 yds: 9.0 to 1. 80 yds: 12.7 to 1.

2) The groups show that high misses are higher than low misses are low.

Conclusions:

1. The ratio of hunters' vertical misses to horizontal misses is probably much greater than most bowhunters realize. Every hunter should study Figure 16-1

2. The bowhunter should spend relatively more effort practicing range estimation than he should shooting arrows.

3. The bowhunter should use a rangefinder while hunting. Any shot of over 50 yards is a very "low percentage" shot unless the range is known.

4. A vertical target, such as a frontal view of a head-up deer, is a big help in overcoming range estimation errors. Although some hunters don't like neck shots, the author took great satisfaction in dropping a goat with a shot which severed the third neck vertebra. Higher or lower would also have severed vertebrae. Lower yet would have been through the main body cavity. Slightly left or right would have cut jugular vanes.

5. A horizontal target, such as a squirrel on all-fours, is hard to hit. The same squirrel when standing or climbing is not such a hard target to hit.

6. The "instinctive" shooter probably has an advantage over the sight shooter in the hunting environment. He is estimating distance as he draws and aims, but his estimate need not be translated into numbers. The pin-sight shooter needs to take several steps, which are:

a) Estimate the distance as being "that far".
b) Translate "that far" into numbers.
c) Find the pin sight corresponding to the translated number.
d) Put the pin on the target.

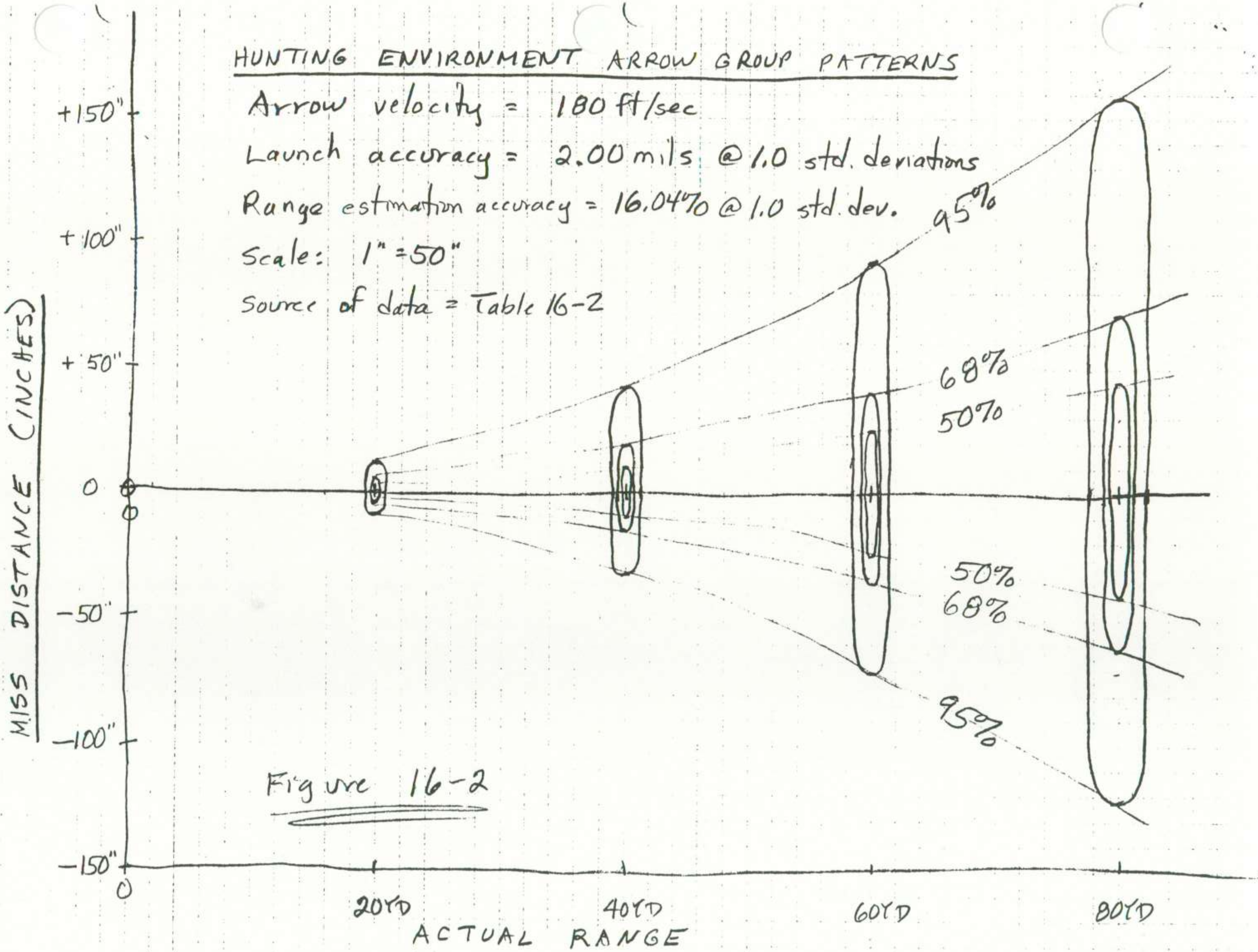


Figure 16-2 ... HUNTING ENVIRONMENT ARROW GROUP PATTERS

Chapter 17 ... CALIBRATED FLIGHT SHOOTING

MAXIMUM RANGE vs TERMINAL VELOCITY vs LAUNCH VELOCITY

The maximum range that an arrow can be shot at sea level is published in this chapter. As far as I know, this is the first time this data has been computed and published. As soon as I make a statement like that, someone will show that the ancient Turks knew all about such things or that the U.S. Army has been teaching such since before the Civil War. Fine, but at least the work published hereinafter is original.

TERMINAL VELOCITIES

An arrow's free-fall terminal velocity is the steady-state speed that it will attain if dropped from an airplane. The terminal velocity is the most meaningful number to describe an arrow's persistence (or lack of it) in traveling through the air. It includes the arrow's weight and the arrow's resistance. A heavy arrow will fall faster than a light arrow if their exteriors are identical. Thus two seemingly identical arrows which do not weigh the same will have different terminal velocities. The heavy arrow will fall faster. As an example, the author converted his bow from standard length to "overdraw". The old, standard-length arrows were 2117 x 30" before I shortened them to 27". New arrows were 2114 x 27". The new 2114 arrows look identical to the old but shortened 2117 arrows and do have identical air resistance characteristics. Yet the 2117 weighs more than the 2114 because it's walls are 17/1000" thick compared to the 14/1000" thick for the 2114. Terminal velocities for the two arrows are roughly:

2117 x 27" w (4) 4" fletches ... V,t = 220 ft/sec
2114 x 27" w (4) 4" fletches ... V,t = 209 ft/sec

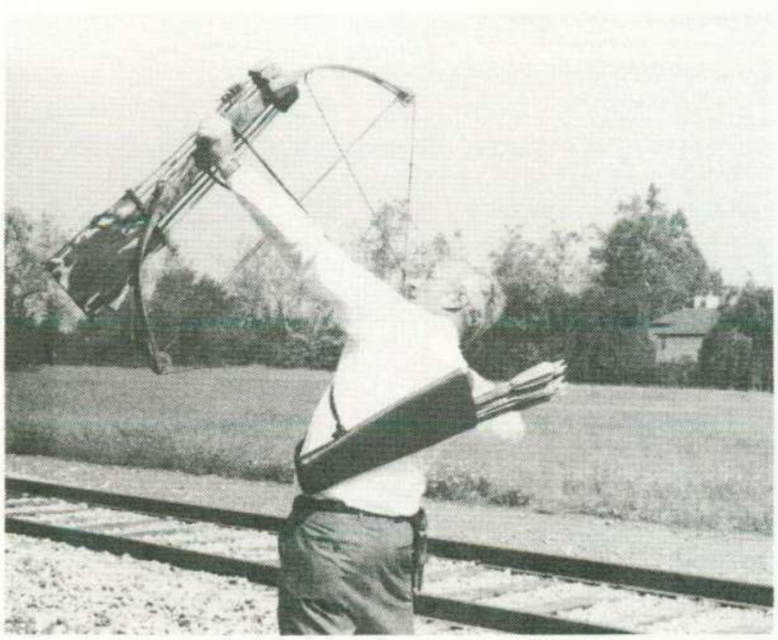
It should be obvious that if both arrows were launched at the same speed and the same angle, the heavier arrow would go farther.

By the same logic, two arrows which weigh the same but which have different friction will fall at different speeds, the more streamlined falling faster. As an example, two 2117 x 30" arrows having different amounts of fletching will have terminal velocities approximately as follows:

2117 x 30" with (3) 5" fletches ... V,t = 230 fps
2117 x 30" with (3) 3" fletches ... V,t = 245 fps

Author shooting a 45# take-down all-fiberglass recurve.

Author shooting his 55# target compound, a Martin Cougar.



Author shooting his 72# hunting compound, a Hoyt Rambo.

DETERMINATION OF TERMINAL VELOCITY

How do you ascertain an arrow's terminal velocity? The most straightforward way is to shoot it as far as it will go. Ascertain launch velocity from an arrow speed measuring device at the local archery shop. Then enter the chart published in this chapter with maximum range and launch velocity. Read terminal velocity.

The next way is to compute the arrow's resistance to air flow as described in the chapter on air resistance. Then weigh the arrow. Then use the following formula to compute terminal velocity:

$$V_{,t} = V_{,o}\,(W/D)^{1/1.85}.$$

Example: Arrow weighs 541 grains.
Calculated drag at 200 ft/sec is 418 grains.

$$V_{,t} = 200 \text{ fps} \times (541/418)^{1/1.85} = 200 \times 1.294^{0.541}$$
$$= 200 \times 1.150 = 230 \text{ fps}.$$

The truly correct way to measure an arrow's terminal velocity is to drop it out of an airplane and measure it's speed upon hitting the ground. Good luck. It is just because taking such a measurement is not practical that I've published this chapter on how to ascertain the arrow's flight characteristics by seeing how far it can be shot.

RANGE OF TERMINAL VELOCITIES:

The slowest arrow will be one with flu-flu fletching. I fired a 2117 x 30" arrow having six 4" long x 1" tall feathers from my 73# compound at 212 fps. The arrow went about 120 yards. Entering the chart yielded the resulting terminal velocity of 96 ft/sec (= 65 mph). Others might put on much more fluffy fletching, so a lower limit of half that seems reasonable. Assume the slowest terminal velocity of any real arrow will be about 50 ft/sec (or 34 mph).

The fastest arrow will be a heavy, thin, short arrow with zero fletching. Admiral Moffett tested just such an arrow in a wind tunnel in the 1920's and found the resistance, without fletching, was 0.016 pounds. This equals 112 grains. Test speed was 200 fps. Length was 26". Diameter was 5/16". A heavy arrow of that size would weigh about 530 grains. Thus the terminal velocity for this arrow would be:

$$V_{,terminal} = V_{,test}\,(W/D)^{(1/1.85)} = 200 \times (530/112)^{0.54} = 463 \text{ fps} = 316 \text{ mph}.$$

Thus I assumed that the fastest terminal velocity in real arrows would not exceed 500 fps.
The range of terminal velocities for real arrows was thus assumed to be 50 to 500 ft/sec, or 34 to 341 mph.

RANGE OF LAUNCH SPEEDS

The fastest launch speed to which I found reference in any of my readings was in Hoyt/Easton's bow advertisement wherein they said, "Bob Rhode used a pair of Contender limbs to power his 400 FPS plus experimental Hoyt/Easton Flight Bow to set two new world records of 845 yards in regular compound flight and 469 yards in Broadhead compound flight in the 70 lb. weight class at the 1986 N.A.A. Flight Championships at Wendover, Utah." Thus 450 fps was selected as the fastest launch.

DRAG-TO-WEIGHT RATIO

The ratio of an arrow's weight to it's drag is the ratio which initially seems most pertinent. The drag-to-weight ratio and the launch-velocity-to-terminal-velocity ratio are really telling the same story. The two ratios are related as follows:

$$W/D = (V_{,t}/V_{,o})^{1.85}$$

where W = weight
D = drag
V,t = velocity, terminal
V,o = velocity, launch.

I chose to work with velocities because the terminal velocity is such an independent quantity. When describing a 2117 x 30 with (3) 5" fletches, I found myself computing a whole slew of different drag values (depending upon launch speed) and then trying to remember what launch speed was associated with the drag value so calculated. It is much easier to simply remember that the arrow's terminal velocity is, for instance, 230 ft/sec. The arrow's terminal velocity stays the same regardless of how the arrow is used or from which bow it is shot. An arrow's terminal velocity stays the same regardless of launch angle or launch speed.

LAUNCH ANGLE

A launch angle of 45° would achieve the greatest range in the vacuum of the moon. Real arrows fired on earth at sea level achieve maximum range using angles of less than 45°. How much less? It depends upon how much drag the arrow has and upon the launch speed. The "dirtiest" arrow computed is one having a terminal velocity of 50 ft/sec, and the highest launch speed computed is 450 ft/sec. This combination has a launch velocity to terminal velocity ratio of 450/50 = 9; and the optimum launch angle for that combination is 25°. Most arrows are fired with Vo/Vt ratios of 0.6 to 1.1. The optimum launch angle for Vo/Vt = 0.6 is 42°. For 1.1 it is about 40°. Actually, the range achieved is very non-sensitive to precise launch angle. In a friction-free environment, the mathematics show that the change of range with change of launch angle is zero at 45°. In the real world with friction, the same is nearly true.

The general idea is that the "cleaner" an arrow is, the closer to 45° it should be shot. Ratios count, too, however. The faster an arrow is shot, the lower it's optimum launch angle becomes. For instance, a flight arrow having a terminal velocity of 300 ft/sec should be launched at 42° if released at 200 ft/sec but should be launched at 38° if released at 400 ft/sec.

CALCULATION OF THE DATA

The formulas for calculation of the trajectories of an arrow with friction and with large launch angles was developed in the last half of Chapter 8. Figure 8-1 shows a typical computer run output for a single combination of launch velocity and terminal velocity. Launch angles of 39°, 40°, 41° & 42° were needed in order to ascertain which of those launch angles yielded the maximum range. While it was at it, the computer kicked out the initial drag-to-weight ratio, the hit-the-ground angles, velocities and energies, plus the time of flight. A separate set of runs such as shown in Figure 8-1 were made for every combination of launch speed and terminal velocity.

THE DATA

Table 17-1, "Table of Maximum Ranges and Corresponding Launch Angles" shows (across the top) terminal velocities from 50 fps to 500 fps plus one column for an infinite terminal velocity. Down the left are launch velocities from 125 ft/sec to 450 ft/sec. In the table, for any combination of launch and terminal velocities, are given the maximum range and corresponding launch angle.

Example #1: Launch velocity and range are known; find terminal velocity.

Launch velocity: V,o = 175 ft/sec.
Range achieved: R,max = 229.0 yards.

Go down to V,o = 175 and then right until R,max = 229.0 yards; thence up to find V,t = 250 ft/sec. Note that optimum angle for that shot would have been 41.6°.

Example #2: Terminal velocity and range achieved are known; find launch velocity.

Terminal velocity = 300 ft/sec.
Range achieved = 364.7 yards.

Go horizontally to V,t = 300 ft/sec. Then down to R,max = 364.7; thence to the left to find V,o = 225 ft/sec = launch velocity. Note that optimum launch angle was 41.3°.

PLOT OF THE DATA

In Figure 17-1, the data is plotted, with launch velocities on the y-axis and maximum ranges on the x-axis, plotted against arrow's terminal velocity.

Example: Launch velocity = 200 ft/sec.
Maximum range = 260 yards.

Enter chart at 200 fps on left and draw a horizontal line.
Enter chart at 260 yards on bottom and draw a vertical line.
Intersection shows arrow's terminal velocity was 226 ft/sec.

Table 17-1 ... MAXIMUM RANGES (yards) & CORRESPONDING LAUNCH ANGLES (degrees)

TERMINAL VELOCITIES =		50fps	75	100	150	200	250	300	350	400	450	500	Zero drag shot
Arrows, typical =			Flu-flu			Broad-head	2117x30 w/(3)5"	2117x30 w/(3)3"	Flight arrow	< Extremely slick >			
LAUNCH VELOCITY 125	Range =	38.1	60.3	79.0	106.2	123.0	133.5	140.3	144.9	148.0	150.7	152.8	161.7
	Angle =	33.5	36.8	38.8	41.0	41.8	43.0	43.5	44.5	44.6	44.7	44.9	45.0
150	Range =	43.5	71.4	96.6	135.7	162.1	179.4	192.0	200.0	207.0	211.0	215.7	232.9
	Angle =	32.0	35.0	37.5	40.0	41.2	42.4	43.1	43.5	44.1	44.4	44.7	45.0
175	Range =	48.3	81.0	112.3	164.2	201.0	229.0	247.0	261.3	271.5	279.2	285.3	317.0
	Angle =	30.5	34.0	36.5	39.1	40.7	41.6	42.4	43.0	43.5	44.0	44.5	45.0
200	Range =	52.5	90.3	127.0	191.0	241.3	278.0	306.0	326.0	341.0	353.1	362.0	414.1
	Angle =	29.5	33.0	35.3	38.3	40.0	41.0	41.8	42.7	43.0	43.5	43.7	45.0
225	Range =	56.3	98.4	140.6	217.3	279.4	327.6	364.7	393.1	414.3	431.9	445.6	524.1
	Angle =	28.7	32.0	34.2	37.6	39.4	40.5	41.3	42.0	42.6	43.0	43.5	45.0
250	Range =	59.8	104.0	152.0	241.0	316.9	377.0	424.0	462.0	491.0	515.0	533.2	647.0
	Angle =	28.0	31.0	33.5	36.8	38.7	40.0	40.9	41.6	42.0	42.9	43.0	45.0
300	Range =	65.9	119.0	175.2	286.3	386.7	471.6	542.5	600.8	648.5	687.1	719.4	931.7
	Angle =	27.0	29.5	32.0	35.3	37.5	39.0	40.0	40.8	41.2	41.7	42.2	45.0
350	Range =	71.1	128.0	194.7	325.0	449.0	561.0	657.0	737.0	807.0	865.0	915.0	1268.1
	Angle =	26.0	28.4	30.5	34.1	36.5	38.0	39.4	40.0	40.6	41.2	41.6	45.0
400	Range =	75.6	140.4	212.0	362.5	509.9	645.5	766.7	872.4	964.5	1044.5	1111.8	1656.3
	Angle =	25.0	27.8	29.5	33.0	35.3	37.0	38.3	39.4	40.0	40.8	41.0	45.0
450	Range =	79.7	149.3	227.6	395.1	563.6	723.8	870.4	1001.5	1118.7	1221.5	1311.3	2096.3
	Angle =	24.0	27.0	28.8	32.0	34.3	36.2	37.5	38.7	39.4	40.0	40.4	45.0

Assumption: Resistance varies as the 1.85th power of speed.

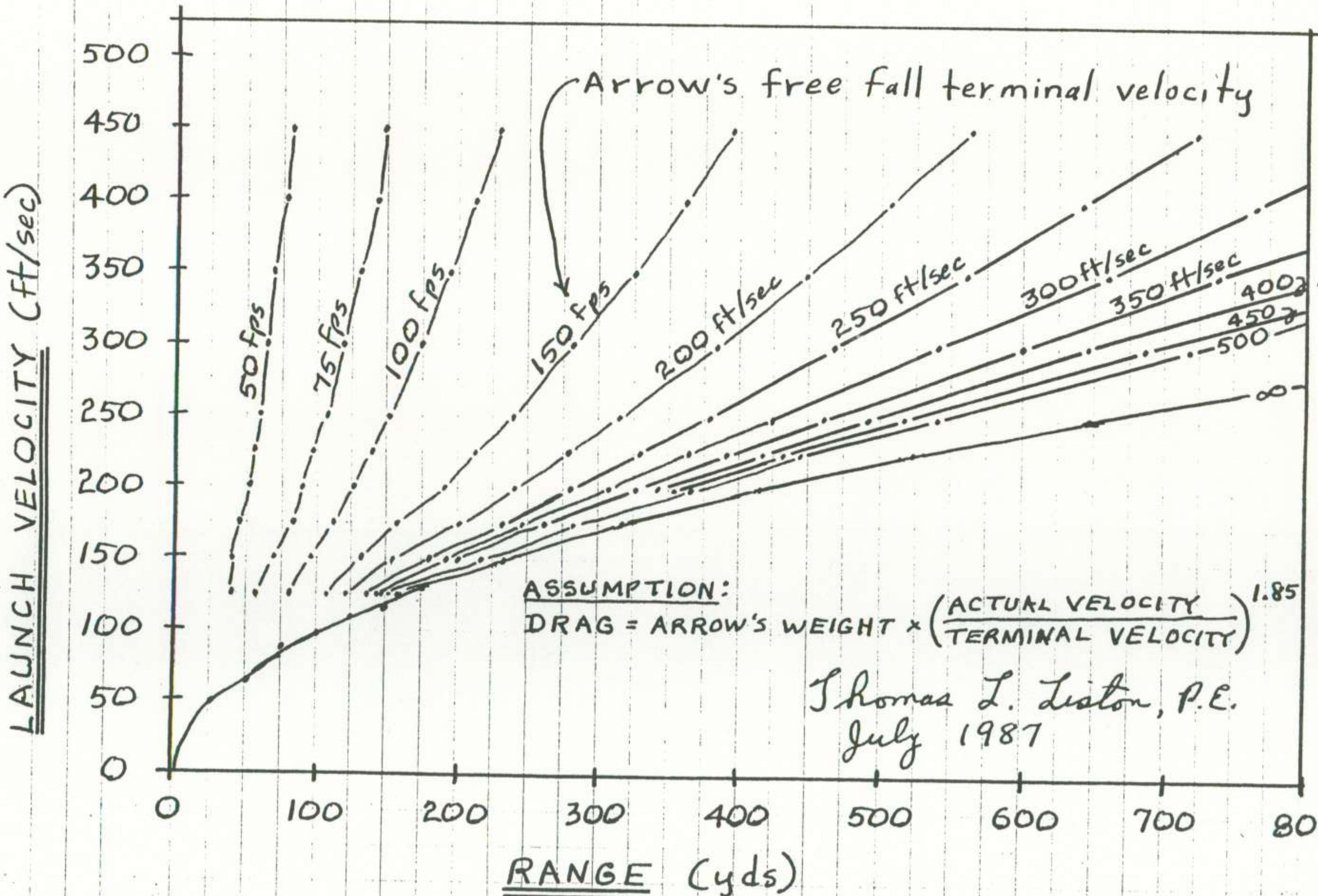


Figure 17-1 ... RANGE vs. LAUNCH VELOCITY for VARIOUS ARROWS

ACCURACY OF THE DATA

The data was computed on the assumption that arrow friction varies as the 1.85th power of velocity. This assumption is subject to review. It may very well be true that the friction of real arrows vary between the 1.5th power and the 2.0nd power, depending upon the arrow. Ultra-steamlined arrows will tend toward the 1.5th power. Flu-flu-fletched arrows will tend toward the 2nd power of speed. Most arrows will be near the 1.85th power.

There was no simple formula for computing the data. Rather, a computer model had to be constructed using the formulas developed in Chapter 8, "Trajectories with Friction", under the section, "Trajectories with Friction and Large Launch Angles". The computer then computed multiple flights at various launch angles. Inspection showed which angle yielded the longest range.

The computer used was a Hewlett-Packard HP-41CV which uses mantissas having 10 digits plus a 2-digit exponent of 10.

Initial runs indicated that breaking the flight up into increments of 0.05 seconds gave data accurate to the nearest yard. Subsequently I found that the flight had to be broken up into 0.02 second increments in order to get answers accurate enough to ascertain the optimum launch angles. The ranges computed with 0.02 second intervals were usually slightly (about 1 yard) longer than those computed at 0.05 second intervals. Even with 0.02 and then 0.01 second intervals, a few combinations came up with seemingly incorrect optimum launch angles. I presume that this is simply an anomaly having to do with the number of digits in the mantissas and the number of increments of the flight. It is possible that a main-frame computer using more digits in each number would give slightly different results.

See the sample computer print-out, Figure 8-1 at the end of Chapter 8.

GENERALIZED DATA

Engineer-scientists always like to find underlying universal parameters to describe things. They are particularly fond of dimensionless parameters. As I studied the data in the table of launch velocity vs terminal velocity with resulting maximum ranges and optimum angles, I noticed that certain ratios were trying to make themselves obvious. I fooled around with various ratios and found to my great pleasure that a single line could replace the multiple lines given in the preceeding chart. The ratios were:

R/R,max = range ratio = maximum range achieved divided by maximum range in a vacuum.

Vo/Vt = velocity ratio = launch velocity divided by arrow's terminal velocity.

See Figure 17-2. The figure is simply a plot of the same data tabulated before, except this time the numbers have been reduced to dimensionless parameters. For instance, in the table is found:

V,o = 200 ft/sec = launch velocity
V,t = 250 ft/sec = terminal velocity
R = 278.0 yards
R, max = 414.1 yards = $(V,o)^2/g$

From which the following can be calculated:

R/R,max = 278.0/414.1 = 0.6713 = range ratio
V,o/V,t = 200/250 = 0.800 = velocity ratio

The data when so reduced is plotted in Figure 17-2, "Range ratio versus velocity ratio". Much to my great pleasure, the data all plotted on a single line!

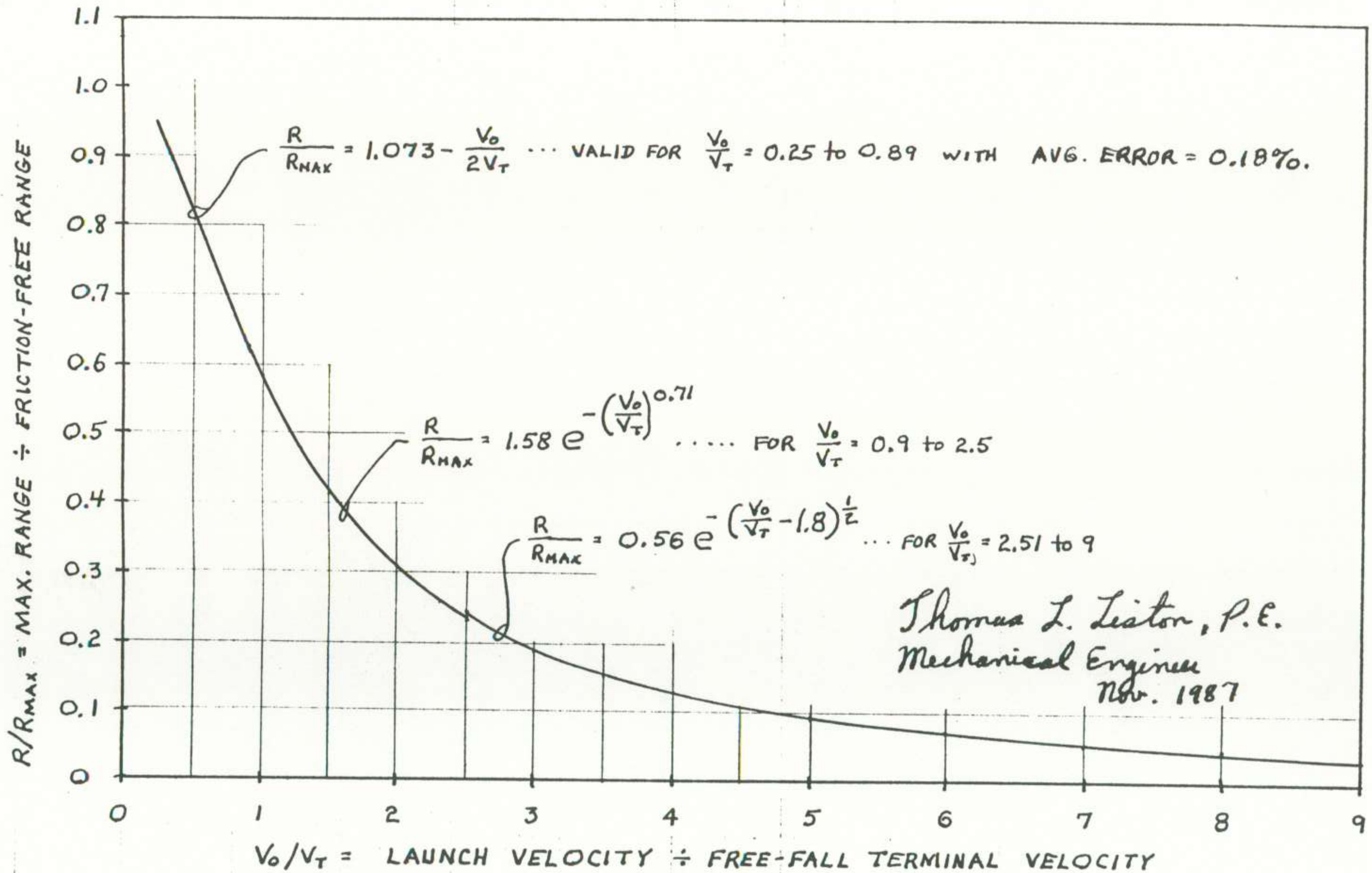


Figure 17-2 ... Range Ratio versus Velocity Ratio

The data shown in Table 17-1 is rearranged and shown again in Tables 17-2A, 17-2B & 17-2C. In addition, data as to the remaining energy, velocity and angle of hitting the ground are shown. The data is arranged according to the ratio of launch velocity to arrow terminal velocity. The "cleanest" shots are first in Table 17.2; the "dirtiest" are last in Table 17-4. The very first entry is for the most slick arrow (terminal velocity = 500 fps) fired at the slowest launch velocity computed (125 fps). That ratio of 125/500 = 0.25 is the closest to a friction-free shot.

"Launch angle" is that angle which the computer found to yield the greatest range. The tabulated launch angles start near 45° and get smaller. That they do not get smaller uniformly is due to computer errors.

"Range ratio" is the ratio of maximum achievable horizontal range divided by the range which an arrow fired at 45° above horizontal without friction would achieve. The friction-free ranges are shown in Table 17-1 in the far right column labeled "zero drag shot". The formula for friction-free range is: $R = V^2/g$ (Eqn #6-4).

"Hit energy" is new data, not shown in Table 17-1. It is the amount of energy the arrow has upon hitting the ground.

Hit velocity is not shown, as it would be redundant. Hit velocity is the square root of hit energy. If the hit energy is 90%, the hit velocity is 94.9% of the launch velocity, because the square root of .900 is .949.
Example: The hit energy of the 125 fps launch, 500 fps terminal arrow is shown to be 88.9%. What is the hit velocity?
Answer: $125 \times (.889)^{1/2} = .943 \times 125 = 118$ fps.

"Hit angle" is the angle at which the arrow hits the ground. They start at -45° (for a "clean" arrow launched at +43.5°) to -72° (for a "dirty" arrow launched at +24°).

"Flight time" is tabulated as a percent of the length of time the same arrow would have stayed airborne in a friction-free environment were it fired at the same angle and with the same launch speed. Note that the comparison is made to an arrow fired at the same angle, not to an arrow fired at +45°. The formula for friction-free flight time is:

$$T = (2V/g) \text{ sine } A. \quad \text{(Equation \#6-5).}$$

The time of flight for the first entry was, then:

$$T = 99\% \times (2 \times 125\text{fps} / 32.2 \text{ fpss}) \text{ sine } 43.5°$$
$$T = .99 \times (250 \text{ sec} /32.2) \times .688 = 5.3 \text{ seconds.}$$

Table 17-2A ..... Tabulation of Computer Model Data for launch-velocity-to-terminal-velocity ratios of 0.25 to 0.889

Vo/Vt ratio	Arrow Vt (fps)	Launch Vo (fps)	Launch Angle (°)	Actual Range (yds)	Range Ratio (%)	Hit energy (%)	Hit angle (°)	Flight Time (%)
.250	500	125	43.5	152.8	94.5%	88.9%	-45°	99%
.278	450	125	45.5	150.9	93.3%	86.9%	-47°	98%
.300	500	150	44.0	215.7	92.6%	85.0%	-46°	98%
.313	400	125	44.0	148.0	91.5%	84.0%	-46°	98%
.333	450	150	43.5	211.0	90.6%	82.0%	-46°	97%
.350	500	175	44.5	285.3	90.0%	81.2%	-47°	97%
.357	350	125	42.5	144.9	89.6%	80.4%	-45°	97%
.375	400	150	43.5	207.0	88.9%	79.0%	-47°	97%
.389	450	175	45.0	279.2	88.1%	78.1%	-48°	97%
.400	500	200	44.0	362.0	87.4%	77.0%	-48°	97%
.417	300	125	44.0	140.0	86.6%	76.0%	-47°	95%
.429	350	150	43.0	200.0	85.9%	75.0%	-47°	96%
.438	400	175	45.0	271.5	85.6%	74.2%	-49°	96%
.444	450	200	43.5	353.1	85.3%	73.4%	-48°	96%
.450	500	225	43.0	445.6	85.0%	72.9%	-47°	96%
.500	350	175	43.0	261.3	82.4%	68.9%	-48°	95%
.500	300	150	43.0	192.0	82.4%	69.0%	-48°	95%
.500	500	250	44.0	533.2	82.4%	69.0%	-49°	95%
.500	400	200	43.0	341.0	82.4%	69.0%	-48°	95%
.500	450	225	43.0	431.0	82.2%	68.9%	-48°	95%
.500	250	125	43.0	133.0	82.2%	68.9%	-48°	95%
.556	450	250	43.0	515.0	79.6%	65.0%	-49°	94%
.563	400	225	43.0	414.3	79.1%	64.1%	-49°	94%
.571	350	200	42.0	326.0	78.7%	63.0%	-48°	94%
.583	300	175	43.0	247.0	77.9%	62.5%	-49°	93%
.600	500	300	43.0	720.0	77.3%	61.3%	-50°	93%
.600	250	150	42.0	179.8	77.2%	61.1%	-49°	93%
.625	200	125	43.0	123.0	76.0%	59.5%	-50°	93%
.625	400	250	43.0	491.0	75.9%	59.0%	-50°	93%
.643	350	225	42.0	393.1	75.0%	58.1%	-49°	93%
.667	300	200	42.0	306.0	73.9%	56.5%	-50°	92%
.667	450	300	42.5	687.1	73.7%	56.6%	-50°	92%
.700	250	175	42.0	229.0	72.2%	54.2%	-50°	91%
.700	500	350	42.0	915.0	72.2%	54.2%	-50°	91%
.714	350	250	42.0	462.0	71.4%	53.0%	-51°	91%
.750	300	225	41.0	364.7	69.5%	50.9%	-50°	91%
.750	200	150	42.0	162.1	69.6%	51.1%	-51°	90%
.750	400	300	42.0	648.5	69.7%	51.1%	-51°	90%
.778	450	350	42.0	865.0	68.2%	49.4%	-52°	90%
.800	250	200	41.0	277.9	67.1%	48.0%	-51°	89%
.800	500	400	41.0	1111.8	67.1%	48.0%	-51°	89%
.833	300	250	41.0	424.0	65.5%	46.1%	-51°	89%
.833	150	125	41.0	106.2	65.7%	46.1%	-52°	89%
.857	350	300	41.5	600.8	64.5%	45.0%	-52°	88%
.875	400	350	40.0	807.0	63.6%	44.0%	-51°	88%
.875	200	175	41.0	201.0	63.4%	43.9%	-52°	88%
.889	450	400	41.0	1044.5	63.1%	43.2%	-52°	88%

Formula: $R/Rmax = 1.073 - Vo/2Vt$
or $R = (V^2/g)(1.073 - Vo/2Vt)$
Formula fits above data with average absolute error = 0.18%.

Table 17-2B .... Tabulation of Computer Model Data for launch-velocity-to-terminal-velocity ratios of 0.9 to 2.5

Vo/Vt ratio	Arrow Vt (fps)	Launch Vo (fps)	Launch Angle (°)	Actual Range (yds)	Range Ratio (%)	Hit energy (%)	Hit angle (°)	Flight Time (%)
.900	500	450	41.0°	1311.3	62.6%	42.7%	-53°	88%
.900	250	225	39.5°	327.6	62.5%	42.4%	-51°	88%
1.000	200	200	40.0°	241.3	58.3%	37.9%	-53°	86%
1.000	300	300	39.0°	542.5	58.1%	37.7%	-52°	86%
1.000	350	350	40.0°	737.0	58.1%	37.9%	-53°	86%
1.000	400	400	39.0°	964.5	58.2%	37.7%	-52°	86%
1.000	150	150	41.0°	135.7	58.3%	38.1%	-54°	86%
1.000	250	250	40.5°	377.0	58.3%	38.0%	-54°	86%
1.000	450	450	40.0°	1221.5	58.3%	37.9%	-53°	86%
1.125	200	225	38.0°	279.4	53.3%	32.6%	-53°	84%
1.125	400	450	39.0°	1118.7	53.4%	32.8%	-54°	84%
1.143	350	400	39.5°	872.4	52.7%	32.3%	-55°	83%
1.167	150	175	38.0°	164.2	51.8%	31.2%	-53°	83%
1.167	300	350	40.0°	657.0	51.8%	32.0%	-56°	83%
1.200	250	300	39.3°	471.6	50.6%	30.4%	-55°	82%
1.250	200	250	39.0°	316.0	48.8%	28.8%	-56°	81%
1.250	100	125	40.0°	79.0	48.8%	29.0%	-57°	81%
1.286	350	450	38.8°	1001.5	47.8%	27.8%	-56°	80%
1.333	150	200	38.0°	191.0	46.1%	26.0%	-55°	80%
1.333	300	400	38.0°	766.7	46.3%	26.3%	-56°	80%
1.400	250	350	38.0°	561.0	44.2%	24.6%	-57°	79%
1.500	150	225	37.0°	217.3	41.5%	22.3%	-57°	77%
1.500	200	300	37.0°	386.7	41.5%	22.4%	-57°	78%
1.500	300	450	37.5°	870.4	41.5%	22.4%	-57°	77%
1.500	100	150	38.0°	96.6	41.5%	22.4%	-58°	77%
1.600	250	400	37.0°	645.5	39.0%	20.3%	-58°	75%
1.667	75	125	36.0°	60.3	37.3%	19.1%	-58°	75%
1.667	150	250	36.0°	241.0	37.2%	19.0%	-58°	74%
1.750	200	350	36.0°	449.0	35.4%	17.7%	-58°	73%
1.750	100	175	36.0°	112.3	35.4%	17.7%	-58°	73%
1.800	250	450	36.5°	723.8	34.5%	17.1%	-60°	72%
2.000	100	200	35.5°	127.0	30.7%	14.5%	-60°	69%
2.000	150	300	34.0°	286.3	30.7%	14.3%	-59°	68%
2.000	200	400	35.0°	509.9	30.8%	14.4%	-60°	70%
2.000	75	150	35.0°	71.4	30.7%	14.4%	-60°	70%
2.250	100	225	34.0°	140.6	26.8%	11.9%	-61°	67%
2.250	200	450	34.0°	563.6	26.9%	11.9%	-61°	67%
2.333	75	175	35.0°	81.0	25.5%	11.3%	-63°	65%
2.333	150	350	35.0°	325.0	25.6%	11.3%	-63°	65%
2.500	100	250	34.0°	152.0	23.5%	10.0%	-63°	62%
2.500	50	125	34.0°	38.1	23.6%	10.0%	-63°	63%

Formula is: $R/Rmax = 1.58 e^{-\left(\frac{V_o}{V_t}\right)^{0.71}}$
Formula fits above data with average absolute error = 0.35%.

Table 17-2C ...... Tabulation of Computer Model Data for launch-velocity-to-terminal-velocity ratios of 2.51 to 9.00

Vo/Vt ratio	Arrow Vt (fps)	Launch Vo (fps)	Launch Angle (°)	Actual Range (yds)	Range Ratio (%)	Hit Energy (%)	Hit Angle (°)	Flight Time (%)
2.667	75	200	33.3	90.3	21.8	9.0	-61.5	62
2.667	150	400	32.0	362.5	21.9	8.9	-61.8	62
3.000	50	150	33.0	43.0	18.5	7.4	-64.9	58
3.000	75	225	33.0	98.4	18.8	7.4	-65.1	58
3.000	100	300	33.0	175.2	18.8	7.4	-65.0	58
3.000	150	450	32.0	395.1	18.8	7.4	-63.9	59
3.333	75	250	31.0	104.0	16.1	6.2	-64.3	55
3.500	50	175	30.5	48.3	15.2	5.6	-64.5	53
3.500	100	350	29.5	194.7	15.4	5.5	-63.7	55
4.000	50	200	30.0	52.5	12.7	4.4	-66.4	51
4.000	75	300	30.0	119.0	12.8	4.4	-66.5	51
4.000	100	400	30.0	212.0	12.8	4.4	-66.5	51
4.500	50	225	29.0	56.3	10.7	3.6	-67.1	48
4.500	100	450	28.0	227.6	10.9	3.5	-66.1	48
4.667	75	350	28.0	128.0	10.1	3.4	-66.5	46
5.000	50	250	28.0	59.8	9.2	2.9	-67.0	45
5.333	75	400	27.0	140.4	8.5	2.6	-67.5	44
6.000	50	300	27.0	65.9	7.1	2.1	-69.1	41
6.000	75	450	26.5	149.3	7.1	2.1	-68.5	41
7.000	50	350	27.0	71.1	5.6	1.6	-71.2	37
8.000	50	400	25.0	75.6	4.6	1.3	-71.6	35
9.000	50	450	24.0	79.7	3.8	1.0	-72.0	33

Formula is: $R/Rmax = 0.56e^{-\left(\frac{V_o}{V_t} - 1.8\right)^{\frac{1}{2}}}$
Formula fits above data with average absolute error = 0.97%.

THE FORMULAS

At the bottom of Tables 17-2A, 17-2B & 17-2C are formulas which fit the range data. There is a different formula for each table. Using the formula, the range data can be computed directly without referring to the tabulated data.

The formulas were found after noting that the data representing all these millions of computer calculations fell on a single line. I looked for a single formula to replace the single line. Much to my amazement, I found that a straight-line formula fit about half of the data perfectly! The straight line formula is applicable for velocity ratios of 0.25 to 0.90. The formula is:

$R/R_{max} = 1.073 - V_o/2V_t$ .... Equation 17-1

Example:

My 2117 x 30" arrows with (3) 5" fletch have terminal velocities of about: V_t = 230 ft/sec. I launch it at about: Vo = 200 ft/sec. Therefore V_o/V_t = 200/230 = 0.870, which is within the formula's range of validity. The range ratio, therefore is:

R/R_{max} = 1.073 - 0.870/2 = 1.073 - 0.435 = 0.638.
$R_{max} = V_o^2/g$ = (200ft/sec)^2/32.2ft/sec^2 = 1242 ft = 414 yards
R = 0.638 x R_{max} = 0.638 x 414 yards = 264 yards.

Thus the predicted range that this bow can shoot this arrow is 264 yards.

The fact of the matter is, the arrow does go 264 yards, and I conclude therefrom that the arrow's terminal velocity is 230 ft/sec.

The curve fits the computer-generated data with accuracy limits of +0.36% to -0.41% and with an average absolute accuracy of 0.18%. See Table 17-2A, "Tabulation of Data for V_o/V_t = 0.25 to 0.89.

The portion of the graph where Vo/Vt is greater than 0.9 is definitely a curve. I fished around for a formula which would fit and finally settled upon two formulas, one for Vo/Vt = 0.9 to 2.5 and one for Vo/Vt = 2.51 to 9.0. The first formula is:

$$R/R_{max} = 1.574\, e^{-\left(\frac{V_o}{V_t}\right)^{0.71}} \quad \text{.... Equation 17-2}$$

where e = 2.71828
valid for V_o/V_t = 0.9 to 2.5

The curve fits the computer-generated data with accuracy limits of +1.00% to -1.22% and with an average absolute accuracy of 0.45%. See Table 17-2B, "Tabulation of Data for Vo/Vt = 0.9 to 2.5."

Similarly, for the range of 2.51 to 9.0, I fished around and settled upon:

$$R/R,max = 0.56e^{-\left(\frac{V_o}{V_t}-1.8\right)^{1/2}} \quad .... \quad \text{Equation 17-3}$$

where e = 2.71828
valid for V,o/V,t = 2.51 to 9.0.

The curve fits the computer-generated data with accuracy limits of +2.33% to -1.25% and with an average absolute accuracy of 1.8%. See Table 17-2C, "Tabulation of Data for Vo/Vt = 2.51 to 9.0."

OPTIMUM LAUNCH ANGLES VERSUS VELOCITY RATIOS

A study of the data and an averaging of it yields the following aproximate relationship between optimum launch angles and velocity ratios:

Vo/Vt =	0.1	0.3	0.5	0.7	0.9	1.0	1.2	1.4	1.6	1.7	2.0	2.4	2.7	3.0
Ao =	45°	44°	43°	42°	41°	40°	39°	38°	37°	36°	35°	34°	33°	32°

Vo/Vt =	3.5	4.0	4.5	5.1	6.0	7.5	8.5	= launch velocity/terminal velocity.
Ao =	31°	30°	29°	28°	27°	26°	25°	= launch angle for maximum range.

Example: I analyzed Bob Rhode's 469 yard broadhead shot at a launch velocity of 400 fps and decided that the arrow's terminal velocity was 187 fps. The ratio is: 400/187 = 2.14. The above chart says, then, that the optimum launch angle for that shot would have been 35°. His flight arrow, which went 845 yards and which therefore had a terminal velocity of 337 fps, would have the ratio 400/337 = 1.19. From the chart, the optimum launch angle would have been 39°. Note that the "cleaner" flight arrow is best fired at closer to 45° than the "dirty" broadhead. "Clean" and "dirty" as used here are taken from aircraft terminology where "dirty" means that flaps and landing gear are down, whereas "clean" means flaps and gear are up.

GRAPHIC DISPLAY OF DATA

Figure 17-3, "Flight Shooting Angles and Ratios", displays all of the data in dimensionless form. The horizontal axis is the velocity ratio. Most archery is done with this ratio near 1.0. To get a feel for what a velocity ratio of 1.0 represents, it is the situation which prevails if you shoot an arrow straight down (as off of Yosemite's Glacier Point) and the arrow neither speeds up nor slows down. If the arrow speeds up after being shot straight down, it's velocity ratio is less than one. If it slows down, it is more than 1.0.

"Launch Angle" is the angle at which the arrow must be launched in order to achieve maximum range.

"Arrival Angle" is the angle at which the arrow will hit the ground.

"Duration of Flight" is how long the arrow stays up compared to an arrow shot in a friction-free environment.

"Arrival velocity" is the speed upon arrival compared to launch speed.

"Range" is actual range compared to an arrow shot at 45° in a friction-free environment.

"Energy" is the energy remaining compared to energy at launch.

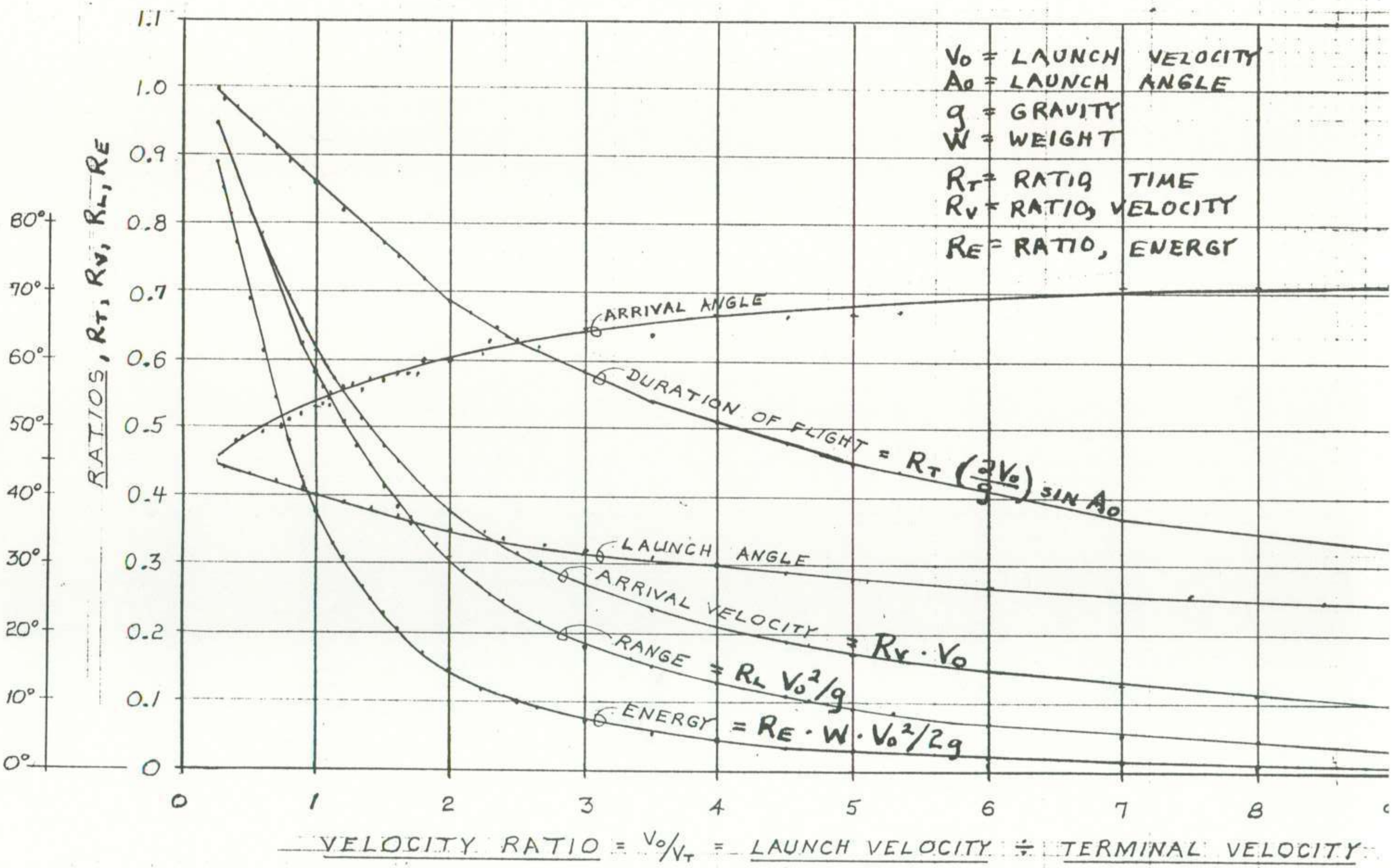


Figure 17-3 ... FLIGHT SHOOTING ANGLES and RATIOS

WIND CORRECTIONS

Wind obviously makes a difference, particularly when shooting for maximum range. The question is, how much difference? Table 17-3, "Wind Corrections to Maximum Range Flight Shots" tabulates the answers for two arrows fired at two velocities. The two arrows where those whose free fall velocities bracketed those of my own hunting arrows. My 2117 x 30" with (3) 5" spiral plastic vanes with field tips have terminal velocities around 230 fps. The arrows chosen had terminal velocities of 200 & 250 fps. The launch velocities were 150 and 200 fps. The results surprised me. The calculations were made for headwinds and tailwinds only. Cross winds were not computed. Results can be summarized as follows:

Conclusions:

Headwind and tailwind corrections are directly proportional to wind speed.
With a launch velocity of 200 fps, range changes about 1½ yard per mph.
With a launch velocity of 150 fps, range changes about 1 yard per mph.

Arrow behavior in wind

At the instant following launch the arrow aligns itself with the relative wind. A new angle of flight is immediately set up which is different than the launch angle, unless there is no wind. Also, immediately a new "airspeed" is established which is faster than launch speed if there is a headwind. If a tailwind, airspeed will be slower than launch speed. The phrase "airspeed" was borrowed from the aviation world. The arrow's flight with respect to the air is exactly as though the arrow were launched in calm air but with the post-alignment angle and post-launch airspeed.

The concept of alignment can be better understood by visualizing shooting downwind 45° above horizontal with such a strong tailwind that wind speed equaled the horizontal component of arrow speed. This would require a 96 mph wind for an arrow launched at 200 fps. The arrow would immediately align itself to aim straight up. The relative wind as experienced by the arrow would be vertical only and equal to the vertical component of launch velocity. To compute this arrow's flight, the flight of a no-wind arrow fired straight up at a velocity equal to launch velocity multiplied by the cosine of 45° would apply. The arrow would return to earth downwind a distance equal to wind speed multiplied by time aloft.

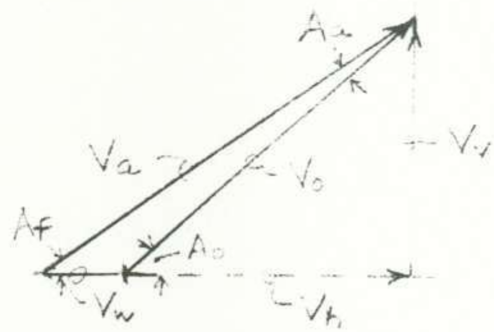
The calculation of the post-launch angles and airspeeds is as shown in Figure 17-4, "Headwind Diagram" and Figure 17-5, "Tailwind Diagram".

WIND CORRECTIONS TO MAXIMUM RANGE FLIGHT SHOTS

Head wind (mph)	Air speed (fps)	Initial angle (deg)	Time of flight (sec)	Travel thru air (yards)	Air movement (yards)	Wind Result (yards)	Total range (yards)	YARDS/ MPH (yd/mph)
Terminal Velocity = 250 fps; Launch velocity = 200 ft/sec. TV/LV = 1.25 Zero friction range = 414 yards								
30	235	35	7.28	339	-107	-46	232	1.5
20	223	37	7.37	321	-72	-29	249	1.5
10	211	39	7.37	299	-36	-15	263	1.5
5	206	41	7.39	288	-18	-8	270	1.6
0	200	42	7.44	278	0	0	278	
-10	189	45	7.45	255	36	13	291	1.3
-20	179	48	7.51	232	73	27	305	1.4
-30	170	52	7.53	207	110	39	317	1.3
-40	161	56	7.57	182	148	52	330	1.3
Terminal Velocity = 200 fps; Launch velocity = 200 ft/sec. TV/LV = 1.0 Zero friction range = 414 yards								
30	236	32	6.55	287	-96	-49	191	1.6
20	224	34	6.62	273	-65	-32	208	1.6
10	212	36	6.64	257	-32	-15	225	1.5
5	206	38	6.70	249	-16	-7	233	1.5
0	200	39		240	0	0	240	
-10	189	42	6.78	224	33	17	257	1.7
-20	178	45	6.79	204	66	30	270	1.5
-30	168	48	6.85	186	100	46	286	1.5
-40	159	52	6.93	166	136	62	302	1.5
Terminal Velocity = 250 fps; Launch velocity = 150 ft/sec. TV/LV = 0.6 Zero friction range = 233 yards								
30	185	34	5.81	238	-85	-26	153	.9
10	161	39	5.89	200	-29	-8	171	.8
0	150	43	5.90	179	0	0	179	
-10	140	47	5.97	159	29	9	188	.9
-30	122	57	5.99	114	88	23	202	.8
Terminal Velocity = 200 fps; Launch velocity = 150 ft/sec. TV/LV = 1.333 Zero friction range = 233 yards								
30	185	32	5.40	209	-79	-32	130	1.1
20	173	35	5.45	194	-53	-21	141	1.1
10	161	38	5.47	178	-27	-11	151	1.1
0	162	41	5.53	162	0	0	162	
-10	139	45	5.55	144	27	9	171	.9
-20	129	50	5.60	126	55	19	181	.9
-30	120	55	5.62	106	82	26	188	.9
-60	102	76	5.74	42	168	48	210	.8
-77	98	89.85	5.73	0	216	54	216	.7

Table 17-3 ... Wind Corrections to Maximum Range Flight Shots

Symbols:



A_a = Angle of adjustment
A_f = Angle of flight
A_o = Angle of launch
V_a = Velocity, airspeed
V_h = Velocity, horizontal
V_o = Velocity, original launch
V_v = Velocity, vertical
V_w = Velocity, wind

Known:

Launch angle = A_o = 42°
Launch velocity = V_o = 200 ft/sec.
Tailwind = V_w = +30 mph x 5280 ft/mile / 3600 sec/hr = 44.0 ft/sec

Find:

Initial airspeed = V_a
Post-alignment launch angle = A_f, &
Amount of angular change to align with airstream = A_a

Solution:

1st: Find vertical component of launch speed:

$V_v = V_o$ sine A_o = 200 ft/sec sine 42° = 200 x 0.669 = 133.8 fps.

2nd: Find horizontal component of launcy velocity:

$V_h = V_o \cos A_o$ = 200 ft/sec cosine 41° = 200 x 0.743 = 148.6 fps.

3rd: Find horizontal component of airspeed:

$V_h + V_w$ = (148.6 + 44.0) fps = 192.6 fps.

4th: Find airspeed:

$V_a^2 = V_y^2 + (V_h + V_w)^2 = 133.8^2 + 192.6^2$ = 17,909 + 37,106 = 55,015
V_a = 235 ft/sec.

5th: Find angle after alignment:

$A_f = \arctan(V_v/(V_h + V_w)$ = arctan (133.8/192.6) = arctan 0.695 = 35°.

6th: Find alignment angle:

$A_a = A_f - A_o$ = 35° - 42° = -7°.

Figure 17-4 ... Headwind Diagram.

Symbols:

A_a = Angle of adjustment
A_f = Angle of flight
A_o = Angle of launch
V_a = Velocity, airspeed
V_h = Velocity, horizontal
V_o = Velocity, original launch
V_v = Velocity, vertical
V_w = Velocity, wind

Known:

Launch angle = $A_o = 42°$
Launch velocity = V_o = 200 ft/sec.
Tailwind = V_w = -40 mph x 5280 ft/mile / 3600 sec/hr = -58.7 ft/sec

Find:

Initial airspeed = V_a
Post-alignment launch angle = A_f, &
Amount of angular change to align with airstream = A_a

Solution:

1st: Find vertical component of launch speed:

$V_v = V_o$ sine A_o = 200 ft/sec sine 42° = 200 x 0.669 = 133.8 fps.

2nd: Find horizontal component of launch velocity:

$V_h = V_o \cos A_o$ = 200 ft/sec cosine 41° = 200 x 0.743 = 148.6 fps.

3rd: Find horizontal component of airspeed:

$V_h + V_w$ = (148.6 - 58.7) fps = 89.9 fps.

4th: Find airspeed:

$V_a^2 = V_v^2 + (V_h + V_w)^2 = 133.8^2 + 89.9^2$ = 17,909 + 8,088 = 25,997
V_a = 161 ft/sec.

5th: Find angle after alignment:

$A_f = \arctan (V_v/(V_h + V_w)$ = arctan (133.8/89.9) = arctan 1.49 = 56°.

6th: Find alignment angle:

$A_a = A_f - A_o$ = 56° - 42° = +14°.

Figure 17-5 ... Tailwind Diagram.

Net effect of tailwind &/or headwind

After computing the post-alignment windspeed and angle, a computer run had to be made for that windspeed and launch angle. The computer run ignores wind. The results indicate how far the arrow would have traveled had there been no wind, which is the same distance it actually travels with respect to the air through which it travels. Then the distance the air mass traveled during the flight is either added or subtracted. The results are tablulated in Table 17-3, "Wind Corrections to Maximum Range Flight Shots".

Launch angle adjustment for headwind or tailwind

The optimum launch angle for maximum range is the same regardless of wind!

My intuition told me that the optimum angle would be lower in a headwind and higher in a tailwind. The computer insisted that the maximum range is achieved with the same launch angle (prior to alignment) regardless of wind. So much for intuition.

Friction-free wind corrections

It is contradictory to talk about wind and zero friction at the same time. Yet the analysis is applicable. The wind is assumed to exist to the extent that it can steer the arrow yet not slow it down. And the arrow's path can be thought of as passing through a mass of friction-free air which transports the arrow. The effect on maximum achievable range is calcualted as follows:

Launch velocity	= 212 ft/sec.
Launch angle	= 45°.
Vertical component = 212 fps x sine 45° = 212 x .707	= 149.9 fps
Horiz. component = 212 fps x cos 45° + 20 mph x 1.47	= 179.2 fps
Airspeed = $(149.9^2 + 179.2^2)^{1/2}$	= 233.7 fps.
Angle = arctan (149.9fps/179.2fps) = arctan 0.836	= 39.9°
Range = (V^2/g) sin (2A) .... Equation #6-9.	
= $(233.7^2/32.2)$ sin (2 x 39.9°)	= 1,668.9 ft
Time aloft = (2V sin A)/g ... Equation #6-5	
= (2 x 233.7fps sin 39.9)/32.2fpss	= 9.3 sec
Distance air moved during flight = 9.3 x 20 x 1.47	= 273.1 ft.
Net distance made good = 1,668.9 ft - 273.1 ft	= 1,395.8 ft.
Zero wind range = V^2/g, Eqn. #6-4 = $212^2/32.2$	= 1,395.8 ft
Loss of range due to headwind	= 0 ft

This result should not surprise us. The only reason to conduct this calculation is to prove that the procedure is correct.

DRAG MEASUREMENTS

Trying to identify the drag of an arrow is frustrating. Admiral Moffett had the benefit of a wind tunnel; yet the data he obtained was subject to a lot of interpretation because a fixed arrow in a wind tunnel is not the same as a free flying arrow. For one thing, the arrow is not free to rotate and thus the fletching's "parachute" drag is much different. For another thing, purposing and/or fishtailing doesn't happen in a wind tunnel. Three ways of testing drag occur to me.

Wind Tunnel

One way would be to build a miniature wind tunnel with upward airflow of adjustable velocity. By adjusting velocity until the test arrow was at "terminal" velocity, the drag would be equal to the arrow's weight. There are lots of problems with this idea, though. The first would be that of keeping the arrow in the center of the airstream. Another would be that the test velocity would be significantly faster than the arrow's typical application speed. Finally, purposing and fishtailing would not prevail.

Arrival Groups

Shooting arrows which are identical except for specific differences, such as blunt tip versus bullet tip, and then measuring the differences in their arrival heights can theoretically be used to back-compute drag differences. The major trouble with this procedure is that a tremendous number of arrows have to be shot before an accurate assessment of the difference in their arrival points can be known with confidence. For instance, I tried shooting 2117 x 30" arrows at 173 fps from 30 yards with blunts and with bullet tips. Predicted difference in arrival height was 1.24". After shooting 160 arrows, no clear pattern developed.

Arrow Meters

Shooting through an arrow velocity measuring device is an excellent procedure. Shoot first at zero range to ascertain launch velocity. Shoot next at as far as reasonable, say 30 to 100 yards, to ascertain arrival velocity. Too far a distance is bad because changes in trajectory elevations complicate the back-calculation of velocity. Too short a distance results in not enough velocity loss. The velocity loss needs to be large compared to the variations in measured velocities. The meter I used varied by about 3 ft/sec in launch velocity. For reasonable accuracy, therefore, it would be necessary to shoot far enough that velocity loss would be at least ten times that much, or 30 fps. Data tabulated in Table 8-1 for a 200 fps arrow shows that 90 yards is about right for reducing velocity to 170 fps. I've not taken data using this procedure yet.

Extreme Range

Shooting arrows as far as they will go can tell a lot about an arrow's drag. The arrow that goes farthest obviously has less drag. Putting a number on that difference is not easy, however. The flight of an arrow shot to maximum range may be different, too, because purposing and fishtailing will not continue throughout the flight.

Launch velocity has to be known. The easiest way to ascertain launch speed is to use an arrow velocity meter. Next choice is a careful calculation based on sight pin spacing.

The lack of any published data on how far a real arrow should travel seemed to me to be quite a vacuum, so I've compiled that data from weeks of computer simulations of arrow flights with friction.

Appendix A ... Bibliography

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